Fibered knot and Fintushel-Stern knot surgered 4-manifold

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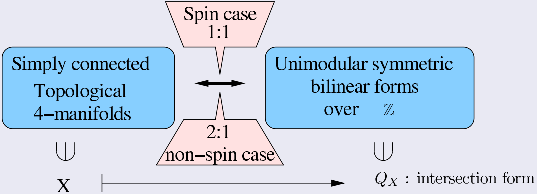
Classification problems in 4-manifolds

Fundamental Questions in Topology and Geometry

- **Existence:** Are there any manifolds with the given properties?
- **Uniqueness:** If there are more than one, then how do we distinguish them?

Freedman(1980) & Donaldson(1982)



Two smooth simply connected closed 4-manifolds are homeomorphic iff they have the same σ, e and type.

Smooth classification of 4-manifolds

For a manifold M^n,

- $n \leq 3$, topological classification is the same as smooth classification.
- $n \geq 5$, every n-manifold has only finitely many distinct smooth n-manifolds which are homeomorphic to it.
- $n = 4$, we have lots of exotic smooth 4-manifolds with the help of Seiberg-Witten invariants. Moreover, all known exotic 4-manifolds has infinitely many different smooth structures.
  Examples: $m\mathbb{CP}^2 \sharp n\overline{\mathbb{CP}}^2$ with $m = 1$ & $n = 2,3,4,5,6,7,8,9$ or $m = 3$ & $n = 4,5,6,7,8,\cdots$ (ABBKP(2007), Park-Yun(2007))

Conjecture (Wild conjecture)

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

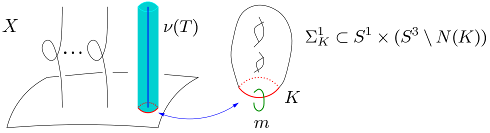
Fintushel-Stern knot surgered 4-manifold

Definition (Fintushel-Stern knot surgered 4-manifold)

X: a closed smooth 4-manifold, $K \subset S^3$: a knot
$\exists T^2 \hookrightarrow X$ with $[T]^2 = 0$
Fintushel-Stern knot surgered 4-manifold is defined by

$$X_K = X \sharp_{T=T_m} (S^1 \times M_K) = (X \setminus (T \times D^2)) \cup_\phi S^1 \times (S^3 \setminus N(K))$$

where $[pt \times \partial D^2]$ is identified with the longitude of K.



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Theorem (Fintushel-Stern, Invent. Math(1998))

1. *if* $b^+(X) > 1$, *then* $\mathscr{SW}_{X_K} = \mathscr{SW}_X \cdot \Delta_K(t)$
2. *if* $b^+(X) = 1$, *then the* $[T]^\perp$*-restricted Seiberg-Witten invariants of* X_K *are* $\mathscr{SW}^{\pm}_{X_K,T} = \mathscr{SW}^{\pm}_{X,T} \cdot \Delta_K(t)$

Note: There are infinitely many inequivalent knots with the same Alexander polynomial.

Motivation

Conjecture (Fintushel-Stern, ICM 1998)

The manifolds $E(2)_{K_1}$ and $E(2)_{K_2}$ are diffeomorphic if and only if K_1 and K_2 are equivalent knots (up to mirror).

- If $K \subset S^3$ is a fibered knot, then $E(n)_K$ has a symplectic structure.
- Moreover **Fintushel-Stern 2004** showed that $E(n)_K$ has a Lefschetz fibration with generic fiber $\Sigma_{2g(K)+n-1}$.

Therefore we are interested in

- explicit monodromy factorization of $E(n)_K$
- Is it unique? I mean, are there nonisomorphic monodromy factorizations on $E(n)_K$ corresponding to fixed generic fiber $\Sigma_{2g(K)+n-1}$?
- some applications?

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Lefschetz fibration

Definition

X^4: compact connected oriented smooth 4-manifold
B:compact connected oriented surface
$\pi : X \to B$, $\pi^{-1}(\partial B) = \partial X$, is a Lefschetz fibration if

- *$\exists C = \{p_1, p_2, \cdots, p_n\} \subset int(X)$: set of critical points of π s.t. $C \neq \varnothing$ & $\pi|_C$ is injective*
- *about each p_i and $b_i := \pi(p_i)$, there are local complex coordinate charts agreeing with the orientations of X and B such that $\pi(z_1, z_2) = z_1^2 + z_2^2$.*

Remark

For a given Lefschetz fibration over S^2 with generic fiber F, we can find an ordered sequence $t_{c_n} \cdot t_{c_{n-1}} \cdot \cdots \cdot t_{c_2} \cdot t_{c_1}$ of right-handed Dehn twists, so called monodromy factorization such that $1 = t_{c_n} \circ t_{c_{n-1}} \circ \cdots \circ t_{c_2} \circ t_{c_1} \in \mathcal{M}_F$.

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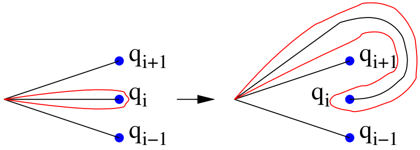
   - Hurwitz moves:
     $t_{c_n} \cdot \ldots \cdot t_{c_{i+1}} \cdot t_{c_i} \cdot \ldots \cdot t_{c_1} \sim t_{c_n} \cdot \ldots \cdot t_{c_{i+1}}(t_{c_i}) \cdot t_{c_{i+1}} \cdot \ldots \cdot t_{c_1}$
   - inverse Hurwitz moves:
     $t_{c_n} \cdot \ldots \cdot t_{c_{i+1}} \cdot t_{c_i} \cdot \ldots \cdot t_{c_1} \sim t_{c_n} \cdot \ldots \cdot t_{c_i} \cdot t_{c_i}^{-1}(t_{c_{i+1}}) \cdot \ldots \cdot t_{c_1}$



Equivalence relations on Lefschetz Fibration

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2. Simultaneous conjugation equivalence of two monodromy factorization is given by

$$t_{c_n} \cdot t_{c_{n-1}} \cdot \ldots \cdot t_{c_2} \cdot t_{c_1} \equiv f(t_{c_n}) \cdot f(t_{c_{n-1}}) \cdot \ldots \cdot f(t_{c_2}) \cdot f(t_{c_1})$$

for some $f \in \mathcal{M}_g$. We will consider $f(w_k \cdot \ldots \cdot w_2 \cdot w_1)$ as $f(w_k) \cdot \ldots \cdot f(w_2) \cdot f(w_1)$

Equivalence relations on Lefschetz Fibration

Theorem (Y.Matsumoto)

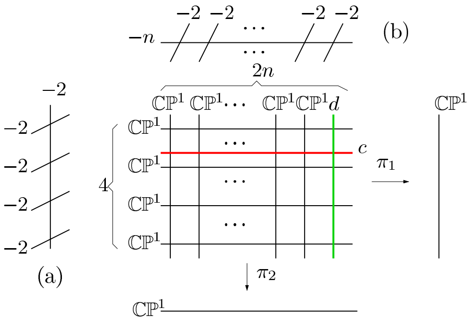
Let $f_i : X_i \to S^2$, $i = 1, 2$, be two Lefschetz fibrations of genus $g \geq 2$. Then the two Lefschetz fibrations are isomorphic if and only if their monodromy factorizations are related by a finite sequence of Hurwitz equivalences and simultaneous conjugation equivalences.

Remark

Lefschetz fibrations $f : M \to B, f' : M' \to B'$ are isomorphic if $\exists$ orientation preserving diffeomorphisms $H : M \to M'$, $h : B \to B'$

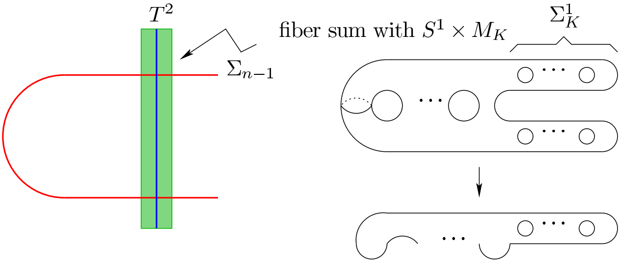
$$\text{such that} \quad \begin{array}{ccc} M & \xrightarrow{H} & M' \\ f\downarrow & & \downarrow f' \\ B & \xrightarrow{h} & B' \end{array} \quad \text{commutes.}$$

Lefschetz fibration structures of $E(n)_K$



- If we consider $\pi_2 : E(n) \to \mathbb{CP}^1$, then generic fiber is T^2 and we have $12n$ nodal type singular fibers with monodromy factorization $(ab)^{6n}$
- If we consider $\pi_1 : E(n) \to \mathbb{CP}^1$, then generic fiber is Σ_{n-1} and we have $8(2n-1)$ nodal type singular fibers with monodromy factorization ι_{n-1}^4 where ι_{n-1} is the hyperelliptic involution of Σ_{n-1}.

Fintushel-Stern's idea:



Therefore we define

Definition

Let $M(n,g)$ be the desingularization of the double cover of $\Sigma_g \times S^2$ branched over $2n(\{pt.\} \times S^2) \cup 2(\Sigma_g \times \{pt.\})$.

Then $E(n)_K$ can be considered as a twisted fiber sum of two $M(n,g)$'s.

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Then $E(n)_K$ can be considered as a twisted fiber sum of two $M(n,g)$'s.

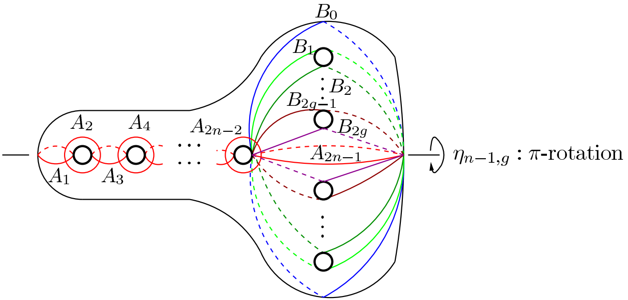
Theorem (Fintushel-Stern, 2004)

$E(n)_K$ can be considered as a twisted fiber sum $M(n,g)\sharp_{\Phi_K}M(n,g)$ by using the diffeomorphism

$$\Phi_K = \varphi_K \oplus id \oplus id : \Sigma_g\sharp\Sigma_{n-1}\sharp\Sigma_g \to \Sigma_g\sharp\Sigma_{n-1}\sharp\Sigma_g$$

where φ_K is a geometric monodromy of fibred knot K.

Lefschetz fibration structures of $E(n)_K$



Theorem (Yun, TAIA(2006))

$\eta^2_{n-1,g}$ is a monodromy factorization of $M(n,g)$ where

$$\eta_{n-1,g} \cong t_{A_{2n-2}} \cdot t_{A_{2n-3}} \cdot \ldots \cdot t_{A_2} \cdot t^2_{A_1} \cdot t_{A_2} \cdot \ldots \cdot t_{A_{2n-2}} \cdot t_{B_0} \cdot t_{B_1} \cdot \ldots \cdot t_{B_{2g}} \cdot t_{A_{2n-1}}$$

Theorem (Yun, TAMS (in press))

Let $K \subset S^3$ be a fibered knot of genus g. Then $E(n)_K$ has a monodromy factorization of the form

$$\Phi_K(\eta_{n-1,g}) \cdot \Phi_K(\eta_{n-1,g}) \cdot \eta_{n-1,g} \cdot \eta_{n-1,g}$$

where $\eta_{n-1,g}$ as before and

$$\Phi_K = \varphi_K \oplus id \oplus id : \Sigma_g \sharp \Sigma_{n-1} \sharp \Sigma_g \to \Sigma_g \sharp \Sigma_{n-1} \sharp \Sigma_g$$

by using a (geometric) monodromy φ_K of the fibered knot K such that

$$S^3 \setminus \nu(K) = (I \times \Sigma_g^1)/((1,x) \sim (0, \varphi_K(x)))$$

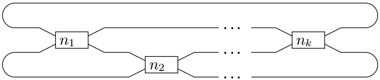
Definition

A 2-bridge knot $b(\alpha, \beta)$ is the knot of the form

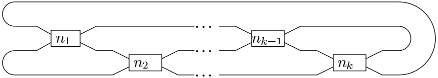
$$C(n_1, -n_2, n_3, -n_4, \cdots, (-1)^{k-1} n_k)$$

where

$$\frac{\beta}{\alpha} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots \cfrac{1}{n_{k-1} + \cfrac{1}{n_k}}}}} = [n_1, n_2, \cdots, n_k].$$



(a) 2-bridge knot with k is odd



(b) 2-bridge knot with k is even

Theorem (Yun, TAMS)

Let K_i, K_j be two 2-bridge knots of the form

$$C(2\varepsilon_{i,1}, 2\varepsilon_{i,2}, \cdots, 2\varepsilon_{i,2g-1}, 2\varepsilon_{i,2g})$$

where $g \geq 1$ and $\varepsilon_{i,k} = +1$ or $\varepsilon_{i,k} = -1$ for each $k = 1, 2, \cdots, 2g$. Let K_0 be the 2-bridge knot of the form $C(\underbrace{-2, -2, \cdots, -2, -2}_{2g})$.

Then for each $n \geq 1$, $E(n)_{K_i} \sharp_{\Sigma_{2g+n-1}} E(n)_{K_0} \approx E(n)_{K_j} \sharp_{\Sigma_{2g+n-1}} E(n)_{K_0}$.

Why is this interesting?

In 2004, Fintushel-Stern defined a family of 4-manifolds

$$Y(n;K_1,K_2) = E(n)_{K_1} \sharp_{\Sigma_{2g+n-1}} E(n)_{K_2}$$

for two fibered knots K_1 and K_2 of genus g. They also proved that

$$\mathcal{SW}_{Y(n;K_1,K_2)} = t_K + (-1)^n t_K^{-1}.$$

Therefore our theorem say that knot type does not determine the smooth type of $Y(n;K_i,K_0)$.

Theorem (Yun, TAMS)

Let K_1, K_2 be any two genus $g \geq 2$ fibred knots in S^3 and let K_0 be the 2-bridge knot $C(\underbrace{-2,-2,\cdots,-2,-2}_{2g})$. Then

$$E(n)_{K_1}\sharp_{t_{b_2}}E(n)_{K_0} \approx E(n)_{K_2}\sharp_{t_{b_2}}E(n)_{K_0}$$

for each fixed integer $n \geq 1$.

Theorem (Park-Yun, 2007(Submitted))

Let $K_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{4g}} = C(2\varepsilon_1, 2\varepsilon_2,\cdots,2\varepsilon_{4g})$ for some $g \geq 1$ and $\varepsilon_i = \pm 1$, then the knot surgery 4-manifold $E(2n)_{K_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{4g}}}$ has at least two nonisomorphic Lefschetz fibration structures for each n.

Tool 1: Monodromy group

Definition

$f: X \to S^2$ a LF with generic fiber F & $W = w_n \cdot \ldots \cdot w_2 \cdot w_1$ be a monodromy factorization.
Monodromy group $G_F(W) \subseteq MCG_F$ is the subgroup of MCG_F generated by $w_1, w_2, \cdots, w_n$.

Lemma (Auroux)

$W_i = w_{i,n_i} \cdot \ldots \cdot w_{i,2} \cdot w_{i,1}$, $i = 1, 2$, be a sequence of right handed Dehn twists along a simple closed curves in Σ_g such that $\lambda_{W_i} = id$. Suppose $f \in G(W_2)$, then $f(W_1) \cdot W_2 \sim W_1 \cdot W_2$.

Lemma

W, W': MF of a genus g LF.
$W \sim W' \Rightarrow G(W) = G(W')$.
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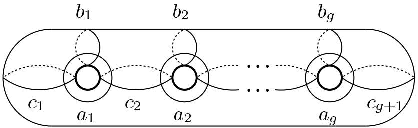
Tool 2: Mapping class group



Theorem (Humphris)

Suppose that $g \geq 2$ and let a_i, b_i, c_i be simple closed curves on Σ_g. Then MCG_g or MCG_g^1 is generated by $t_{c_1}, t_{a_1}, t_{c_2}, t_{a_2}, \cdots, t_{c_g}, t_{a_g}$ and t_{b_2} .

Corollary

Let $g \in \mathbb{Z}_{\geq 3}$.

1. *MCG_g or MCG_g^1 is generated by $\{t_{a_1}, t_{c_2}, t_{a_2}, \cdots, t_{c_g}, t_{a_g}\} \cup \{t_{b_i}, t_{b_{i+1}}\}$ for any $1 \leq i < g$*
2. *MCG_g or MCG_g^1 is not generated by $\{t_{c_1}, t_{a_1}, t_{c_2}, t_{a_2}, \cdots, t_{c_g}, t_{a_g}\} \cup \{t_{b_{2j+1}} \mid 1 \leq j \leq [\frac{g-1}{2}]\}$.*

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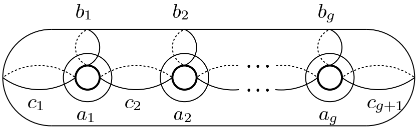

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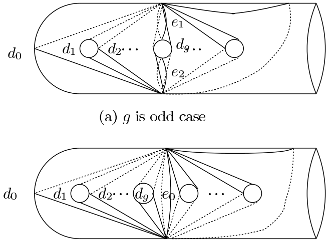
Tool 3: Monodromy map of a fibered knot

Let $K_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{2g}} = C(2\varepsilon_1, 2\varepsilon_2, \cdots, 2\varepsilon_{2g})$ for positive integer $g \geq 2$ and $\varepsilon_i = \pm 1$.

- $\varphi_{K_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{2g}}} = t_{a_g}^{\varepsilon_{2g}} \circ t_{c_g}^{\varepsilon_{2g-1}} \circ \cdots t_{a_1}^{\varepsilon_2} \circ t_{c_1}^{\varepsilon_1}$
- $\phi(\varphi_{K_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{2g}}}) = t_{a_1}^{\varepsilon_{2g}} \circ t_{c_2}^{\varepsilon_{2g-1}} \circ \cdots t_{a_g}^{\varepsilon_2} \circ t_{b_g}^{\varepsilon_1}$

are monodromy of $K_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{2g}}$ where

$$\phi = \begin{cases} t_{d_0} \circ t_{d_1} \circ \cdots \circ t_{d_g} \circ t_{e_0} & , g \text{ is even}, \\ t_{d_0} \circ t_{d_1} \circ \cdots \circ t_{d_g} \circ t_{e_1}^2 \circ t_{e_2}^2 & , g \text{ is odd}. \end{cases}$$



(b) g is even case

Sketch of proof: $E(n)_{K_i}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$

- Let $K_i = C(2\varepsilon_{i,1}, 2\varepsilon_{i,2}, \cdots, 2\varepsilon_{i,2g-1}, 2\varepsilon_{i,2g})$, then $E(n)_{K_i}\sharp_{\Sigma_{n+2g-1}}E(n)_{K_0}$ has a monodromy factorization of the form
$$\Phi_{K_0}(\eta^2_{n-1,g})\cdot\eta^2_{n-1,g}\cdot\Phi_{K_i}(\eta^2_{n-1,g})\cdot\eta^2_{n-1,g}.$$

- We may consider $\Phi_{K_0} = t_{a_g}^{-1}\circ t_{c_g}^{-1}\circ\cdots\circ t_{a_1}^{-1}\circ t_{c_1}^{-1}$ and $\Phi_{K_i} = t_{a_g}^{\varepsilon_{i,2g}}\circ t_{c_g}^{\varepsilon_{i,2g-1}}\circ\cdots\circ t_{a_1}^{\varepsilon_{i,2}}\circ t_{c_1}^{\varepsilon_{i,1}}$.
- $\Phi_{K_i}\in G_F(\Phi_{K_0}(\eta^2_{n-1,g})\cdot\eta^2_{n-1,g})$ because
  - $t_{a_i} = (t_{B_{2i}}^{-1}\circ\Phi_{K_0})(t_{B_{2i}})$
  - $t_{c_i} = (t_{B_{2i-1}}^{-1}\circ\Phi_{K_0})(t_{B_{2i-1}})$

Therefore $E(n)_{K_i}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$ and $E(n)_{K_j}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$ have isomorphic monodromy factorization and it implies diffeomorphism.

- Let $K_i = C(2\varepsilon_{i,1}, 2\varepsilon_{i,2}, \cdots, 2\varepsilon_{i,2g-1}, 2\varepsilon_{i,2g})$, then $E(n)_{K_i}\sharp_{\Sigma_{n+2g-1}}E(n)_{K_0}$ has a monodromy factorization of the form

$$\Phi_{K_0}(\eta^2_{n-1,g})\cdot\eta^2_{n-1,g}\cdot\Phi_{K_i}(\eta^2_{n-1,g})\cdot\eta^2_{n-1,g}.$$

- We may consider $\Phi_{K_0} = t^{-1}_{a_g}\circ t^{-1}_{c_g}\circ\cdots\circ t^{-1}_{a_1}\circ t^{-1}_{c_1}$ and $\Phi_{K_i} = t^{\varepsilon_{i,2g}}_{a_g}\circ t^{\varepsilon_{i,2g-1}}_{c_g}\circ\cdots\circ t^{\varepsilon_{i,2}}_{a_1}\circ t^{\varepsilon_{i,1}}_{c_1}$.
- $\Phi_{K_i}\in G_F(\Phi_{K_0}(\eta^2_{n-1,g})\cdot\eta^2_{n-1,g})$ because
  - $t_{a_i} = (t^{-1}_{B_{2i}}\circ\Phi_{K_0})(t_{B_{2i}})$
  - $t_{c_i} = (t^{-1}_{B_{2i-1}}\circ\Phi_{K_0})(t_{B_{2i-1}})$

Therefore $E(n)_{K_i}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$ and $E(n)_{K_j}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$ have isomorphic monodromy factorization and it implies diffeomorphism.

- Let $K_i = C(2\varepsilon_{i,1}, 2\varepsilon_{i,2}, \cdots, 2\varepsilon_{i,2g-1}, 2\varepsilon_{i,2g})$, then $E(n)_{K_i}\sharp_{\Sigma_{n+2g-1}}E(n)_{K_0}$ has a monodromy factorization of the form
$$\Phi_{K_0}(\eta_{n-1,g}^2)\cdot\eta_{n-1,g}^2\cdot\Phi_{K_i}(\eta_{n-1,g}^2)\cdot\eta_{n-1,g}^2.$$

- We may consider $\Phi_{K_0} = t_{a_g}^{-1}\circ t_{c_g}^{-1}\circ\cdots\circ t_{a_1}^{-1}\circ t_{c_1}^{-1}$ and $\Phi_{K_i} = t_{a_g}^{\varepsilon_{i,2g}}\circ t_{c_g}^{\varepsilon_{i,2g-1}}\circ\cdots\circ t_{a_1}^{\varepsilon_{i,2}}\circ t_{c_1}^{\varepsilon_{i,1}}$.
- $\Phi_{K_i}\in G_F(\Phi_{K_0}(\eta_{n-1,g}^2)\cdot\eta_{n-1,g}^2)$ because
  - $t_{a_i} = (t_{B_{2i}}^{-1}\circ\Phi_{K_0})(t_{B_{2i}})$
  - $t_{c_i} = (t_{B_{2i-1}}^{-1}\circ\Phi_{K_0})(t_{B_{2i-1}})$

Therefore $E(n)_{K_i}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$ and $E(n)_{K_j}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$ have isomorphic monodromy factorization and it implies diffeomorphism.

- Let $K_i = C(2\varepsilon_{i,1}, 2\varepsilon_{i,2}, \cdots, 2\varepsilon_{i,2g-1}, 2\varepsilon_{i,2g})$, then $E(n)_{K_i}\sharp_{\Sigma_{n+2g-1}}E(n)_{K_0}$ has a monodromy factorization of the form
$$\Phi_{K_0}(\eta^2_{n-1,g})\cdot\eta^2_{n-1,g}\cdot\Phi_{K_i}(\eta^2_{n-1,g})\cdot\eta^2_{n-1,g}.$$
- We may consider $\Phi_{K_0} = t_{a_g}^{-1}\circ t_{c_g}^{-1}\circ\cdots\circ t_{a_1}^{-1}\circ t_{c_1}^{-1}$ and $\Phi_{K_i} = t_{a_g}^{\varepsilon_{i,2g}}\circ t_{c_g}^{\varepsilon_{i,2g-1}}\circ\cdots\circ t_{a_1}^{\varepsilon_{i,2}}\circ t_{c_1}^{\varepsilon_{i,1}}$.
- $\Phi_{K_i}\in G_F(\Phi_{K_0}(\eta^2_{n-1,g})\cdot\eta^2_{n-1,g})$ because
  - $t_{a_i} = (t_{B_{2i}}^{-1}\circ\Phi_{K_0})(t_{B_{2i}})$
  - $t_{c_i} = (t_{B_{2i-1}}^{-1}\circ\Phi_{K_0})(t_{B_{2i-1}})$

Therefore $E(n)_{K_i}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$ and $E(n)_{K_j}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$ have isomorphic monodromy factorization and it implies diffeomorphism.

Sketch of proof: $E(n)_{K_i}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$

- Let $K_i = C(2\varepsilon_{i,1}, 2\varepsilon_{i,2}, \cdots, 2\varepsilon_{i,2g-1}, 2\varepsilon_{i,2g})$, then $E(n)_{K_i}\sharp_{\Sigma_{n+2g-1}}E(n)_{K_0}$ has a monodromy factorization of the form
$$\Phi_{K_0}(\eta_{n-1,g}^2)\cdot\eta_{n-1,g}^2\cdot\Phi_{K_i}(\eta_{n-1,g}^2)\cdot\eta_{n-1,g}^2.$$
- We may consider $\Phi_{K_0} = t_{a_g}^{-1}\circ t_{c_g}^{-1}\circ\cdots\circ t_{a_1}^{-1}\circ t_{c_1}^{-1}$ and $\Phi_{K_i} = t_{a_g}^{\varepsilon_{i,2g}}\circ t_{c_g}^{\varepsilon_{i,2g-1}}\circ\cdots\circ t_{a_1}^{\varepsilon_{i,2}}\circ t_{c_1}^{\varepsilon_{i,1}}$.
- $\Phi_{K_i}\in G_F(\Phi_{K_0}(\eta_{n-1,g}^2)\cdot\eta_{n-1,g}^2)$ because
  - $t_{a_i} = (t_{B_{2i}}^{-1}\circ\Phi_{K_0})(t_{B_{2i}})$
  - $t_{c_i} = (t_{B_{2i-1}}^{-1}\circ\Phi_{K_0})(t_{B_{2i-1}})$

Therefore $E(n)_{K_i}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$ and $E(n)_{K_j}\sharp_{\Sigma_{2g+n-1}}E(n)_{K_0}$ have isomorphic monodromy factorization and it implies diffeomorphism.

Sketch of proof: $E(n)_K \sharp_{t_{b_2}} E(n)_{K_0}$

- $t_{a_i}, t_{c_i} \in G(\Phi_{K_0}(\eta^2_{n-1,g}) \cdot \eta^2_{n-1,g})$, $i = 1, 2, \cdots, g$ implies $t_{a_i}, t_{c_i} \in G(t_{b_2}(\Phi_{K_0}(\eta^2_{n-1,g}) \cdot \eta^2_{n-1,g}))$ for all $i = 1, 2, \cdots, g$ except t_{a_2}
- Because $(t_{B_4}^{-1} \circ \Phi_{K_0})(t_{B_4}) = t_{a_2}$ and $t_{B_3}(t_{b_2}(t_{B_3})) = t_{b_2}$,

$$
\begin{aligned}
& t_{b_2}(\Phi_{K_0}(\eta^2_{n-1,g}) \cdot \eta^2_{n-1,g}) \cdot \eta^2_{n-1,g} \\
\sim \quad & t_{b_2}(\Phi_{K_0}(\eta^2_{n-1,g})) \cdot t_{b_2}(\eta_{n-1,g}) \cdot \eta^2_{n-1,g} \cdot t_{b_2}(\eta_{n-1,g}) \\
\sim \quad & \ldots \cdot t_{b_2}(\Phi_{K_0}(t_{B_4})) \cdot \ldots \cdot t_{b_2}(t_{B_4}) \cdot \ldots \cdot t_{B_3} \cdot \ldots \cdot t_{b_2}(t_{B_3}) \cdot \ldots \\
\sim \quad & \ldots \cdot t_{b_2}(t_{B_4}) \cdot t_{b_2}((t_{B_4}^{-1} \circ \Phi_{K_0})(t_{B_4})) \cdot \ldots \cdot t_{B_3}(t_{b_2}(t_{B_3})) \cdot t_{B_3} \cdot \ldots \\
\sim \quad & \ldots \cdot t_{b_2}(t_{a_2}) \cdot \ldots \cdot t_{b_2} \cdot \ldots \\
\sim \quad & \ldots \cdot t_{b_2} \cdot t_{a_2} \cdot \ldots
\end{aligned}
$$

Therefore

$$\Phi_K \in \langle a_i, c_i, b_2 | 1 \leq i \leq g \rangle = MCG^1_g \subseteq G(t_{b_2}(\Phi_{K_0}(\eta^2_{n-1,g}) \cdot \eta^2_{n-1,g}) \cdot \eta^2_{n-1,g})$$

The end. Thank you !