An upper bound for tunnel number of a knot using free genus

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K : knot in S^3

F : Seifert surface of K

= compact connected orientable surface with $\partial F = K$

F is **free** if $\overline{S^3 - N(F)}$ is a handlebody.

$g(K)$: genus of K

= minimal genus among all Seifert surfaces of K

$g_f(K)$: free genus of K

= minimal genus among all free Seifert surfaces of K

$g(K) \leq g_f(K)$

$t(K)$: tunnel number of K

$t(K) = n$ if there are n disjoint properly embedded arcs $t_1, \cdots, t_n$ in $S^3 - K$ such that $\overline{S^3 - N(K \cup \bigcup_{i=1}^n t_i)}$ is a genus $n+1$ handlebody,

and n is a minimum among all such numbers.

Proposition $t(K) \leq 2g_f(K)$

Suppose $g_f(K) = n$. K bounds a once punctured genus n free Seifert surface F.

Take $2n$ disjoint properly embedded arcs $t_1, \cdots, t_{2n}$ on F such that F cut along $\cup_{i=1}^{2n} t_i$ is a disk.

Let D denote the disk $\overline{F - N(K \cup \cup_{i=1}^{2n} t_i\,;\, F)}$.

Take a product neighborhood $N(F) = F \times I$ of F such that $F = F \times \{0\} \subset F \times I$.

$\overline{S^3 - N(F)}$ is a handlebody since F is a free Seifert surface and $D \times I$ can be regarded as a 1-handle attached to it.

So $\overline{S^3 - N(F)} \cup (D \times I)$ is also a handlebody, which is exterior of $K \cup \cup_{i=1}^{2n} t_i$.

$\therefore \quad t(K) \leq 2g_f(K) = 2n$.

When $t(K) < 2g_f(K)$?

When $g_f(K) = 1$, $(1 = g(K) \leq g_f(K))$

Goda-Teragaito conjecture

Theorem (Scharlemann) Suppose $K \subset S^3$ has tunnel number one and genus one. Then either

1. K is a satellite knot or
2. K is a 2-bridge knot.

Attaching (annulus)$\times I$ to handlebody along (annulus)$\times \partial I$

γ_1, γ_2 : disjoint essential loops on the boundary of a handlebody H

D : an essential disk of H such that $|D \cap \gamma_1| = |\partial D \cap \gamma_1| = 1$ and $D \cap \gamma_2 = \emptyset$.

A : an annulus.

Lemma If we attach $A \times I$ to H along $A \times \partial I$ so that $A \times \{0\}$ is attached to $N(\gamma_1\,;\partial H)$ and $A \times \{1\}$ to $N(\gamma_2\,;\partial H)$,

then the resulting manifold is a handlebody of same genus with H.

Notations

F : genus n free Seifert surface for a knot K with $g_f(K) = n$

$t_1, \cdots, t_{2n-1}$: disjoint properly embedded arcs in F such that F cut along $\cup_{i=1}^{2n-1} t_i$ is an annulus A

γ : essential loop of A.

Theorem

Suppose there exists an essential disk D in $\overline{S^3 - (F \times I)}$ such that $|D \cap (\gamma \times \{0\})| = 1$ and $D \cap (\gamma \times \{1\}) = \emptyset$, where $F = F \times \{0\} \subset F \times I$.

Then $t(K) \leq 2g_f(K) - 1$.

Sketch of proof

Remove the collar of ∂A from A.

Obtain $A' = \overline{A - N(\partial A\,;\,A)}$, which is in the interior of F.

The neighborhood of F, $N(F) = F \times I$, can be understood as the union of $N(K \cup \bigcup_{i=1}^{2n-1} t_i)$ and $A' \times I$.

Since F is a genus n free Seifert surface, $\overline{S^3 - (F \times I)}$ is a genus $2n$ handlebody.

$A' \times I$ is attached to it along $A' \times \{0\}$ and $A' \times \{1\}$.

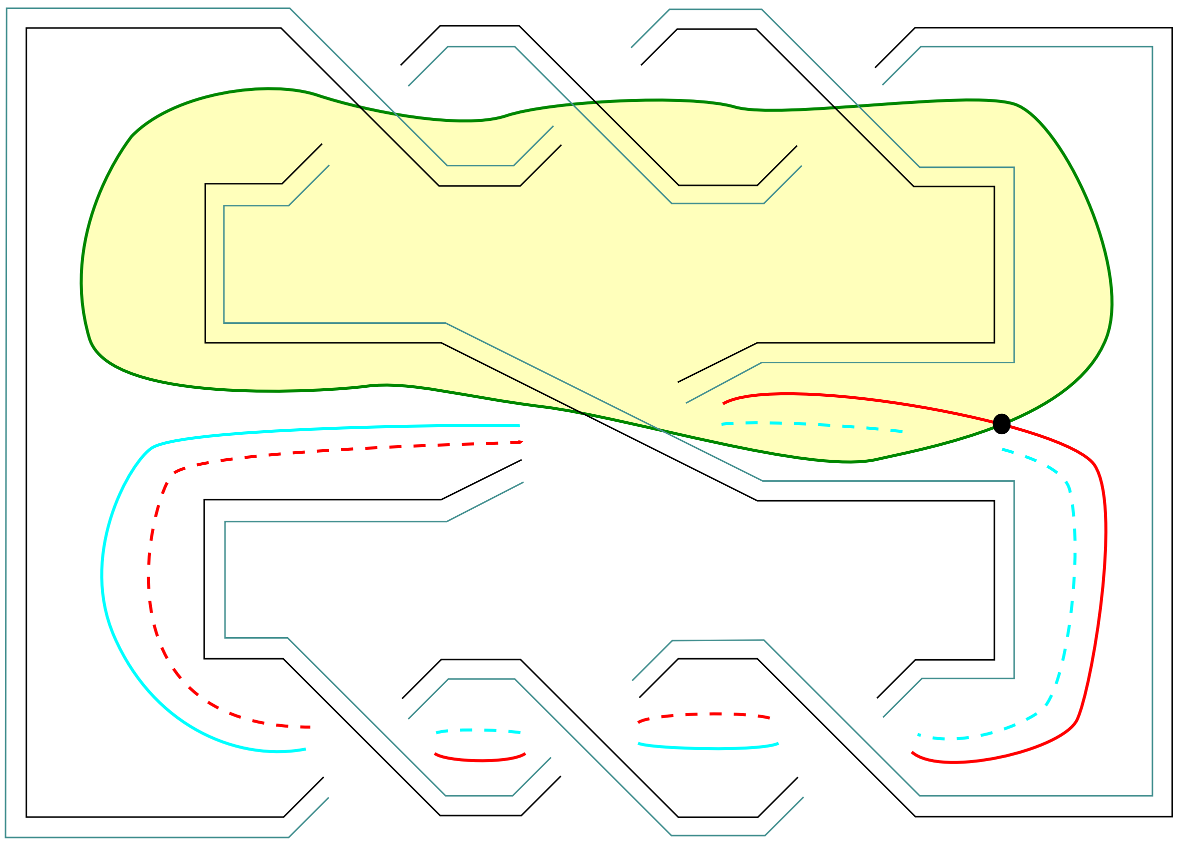
By **Lemma**, $\overline{S^3 - (F \times I)} \cup (A' \times I)$ is a genus $2n$ handlebody, which is the exterior of $K \cup \bigcup_{i=1}^{2n-1} t_i$.

$\therefore \quad t(K) \leq 2n - 1 = 2g_f(K) - 1.$

Examples

Corollary Pretzel knot $P(a_1, a_2, \cdots, a_{2n+1})$ ($a_i =$ odd for all i and $a_i = 1$ for some i) has tunnel number less than or equal to $2n-1$.

The canonical Seifert surface by Seifert algorithm is free.
We can easily find the essential disk D satisfying the condition of **Theorem**.



An example : $P(3,1,3)$

Proposition Let A be an incompressible annulus properly embedded in a handlebody H. Then A cuts H into handlebodies(or a handlebody).

Standard innermost disk and outermost arc argument.

Heegaard splitting of a 3-manifold M is a decomposition $M = H_1 \cup_S H_2$.

(H_1, H_2 are handlebodies and $S = \partial H_1 = \partial H_2$)

From one Heegaard splitting to another

Theorem Let $M = H_1 \cup H_2$ be a Hegaard splitting. Let A be an incompressible annulus properly embedded in H_2 and $\partial A = a_1 \cup a_2$.
Suppose that there exists an essential disk D of H_1 such that $|D \cap a_1| = 1$ and $D \cap a_2 = \emptyset$.

Let H_1' be obtained from H_1 by attaching $A \times I \subset H_2$ along $a_1 \times I$ and $a_2 \times I$.
Let H_2' be obtained from H_2 by cutting along A.

Then $M = H_1' \cup H_2'$ is a Heegaard splitting of same genus with $H_1 \cup H_2$.