On embedding all $n$-manifolds into a single $(n + 1)$-manifold

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2008-01-23
Introduction

Question
Find the smallest nonnegative integer $e_n$, such that any $n$-dimensional connected, closed manifold can be embedded into a single connected, closed manifold of dimension $n + e_n$.

Note
All embeddings are considered to be topologically flat.

- $0 \leq e_n \leq n$, by Whitney embedding theorem,
- $e_0 = e_1 = 0$,
- $e_2 = 1$,
- $e_3 = 2$. 
Problem

Why $e_2 = 1$?

Classification of connected, closed 2-manifolds

- Orientable surface: $nT^2$,
- non-orientable surface: $nT^2 \# \mathbb{R}P^2$ or $nT^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

Lemma

If $M_1^n \hookrightarrow W_1^{n+1}$ and $M_2^n \hookrightarrow W_2^{n+1}$, then

$M_1 \# M_2 \hookrightarrow W_1 \# W_2$.

As $nT^2 \hookrightarrow S^3$, $\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$, every surface embeds in

$S^3 \# \mathbb{R}P^3 \# \mathbb{R}P^3 \cong \mathbb{R}P^3 \# \mathbb{R}P^3$. 
Why $e_3 = 2$?

- Every oriented, closed 3-manifolds embeds into $S^5$. [Hirsch, 1961]
- Every non-orientable, closed 3-manifolds embeds into $S^5$. [Rohlin, 1965; Wall, 1965]
- There does not exist a single oriented, closed 4-manifold such that any connected, closed 3-manifold can be embedded into it. [Kawauchi, 1988]
- The condition “oriented” can be eliminated from the above statement. [Shiomi, 1991]
Does $e_n > 1$?

**Question**

Let $n$ be a positive integer, whether there exists a connected, closed $(n + 1)$-manifold $W$, such that any connected, closed $n$-manifold $M$ can be embedded into $W$.

**Up To Date Results**

- YES for $n = 1, 2,$
- NO for $n = 3$ and $n = 4m - 1$, [Kawauchi].
- Partially YES for $n = 4$,
- No for $n$ is a composite number and is not a power of 2.

We expect the answer to be NO for any $n \geq 4$. 
Theorem

If \( n \) is a composite number and is not a power of 2, then there does not exist a connected, closed \((n + 1)\)-manifold \( W \), such that any smooth, simply-connected, closed \( n \)-manifold \( M \) can be embedded into \( W \).

If such a \( W \) exists and is non-orientable, then its orientation double cover \( \tilde{W} \) will also satisfies the condition, but \( \tilde{W} \) is oriented.

Note

All manifolds are considered to be oriented, connected and closed.
Suppose $M^n \hookrightarrow W^{n+1}$, then $W \setminus M$ could

- be connected
- have two components $W_1$ and $W_2$.

In the latter case, by using the Mayer-Vietoris sequence for $(W_1, W_2)$, we get

**Proposition**

*For any integer factorization $n = pq$, where $p, q > 0$, there exists a subspace $V \subset H^p(M; \mathbb{R})$ such that*

- (ii) $\dim V \geq \frac{1}{2}(\beta_p(M) - \beta_{p+1}(W))$, and
- (iii) for any $x_1, \ldots, x_q \in V$, $x_1 \cup \ldots \cup x_q = 0$. 

Proposition

Suppose $M^n \hookrightarrow W^{n+1}$, then for any integer factorization $n = pq$, where $p, q > 0$, there exists a subspace $V$ of $H^p(M; \mathbb{R})$ and a linear transformation $\varphi : V \rightarrow H^p(M; \mathbb{R})$ such that

(i) $\varphi$ has no fixed non-zero vectors,
(ii) $\dim V \geq \frac{1}{2}(\beta_p(M) - \beta_{p+1}(W))$, and
(iii) for any $x_1, \ldots, x_q \in V$, $x_1 \cup \ldots \cup x_q = \varphi(x_1) \cup \ldots \cup \varphi(x_q)$.

There are 3 items in this statement.

- a subspace
- a linear transformation
- a cup product relation
Let $n = pq$, where $p \geq 2$, $q \geq 3$ and is odd. Given $W^{n+1}$, then $\beta_{p+1}(W)$ is fixed. We will find $M^n$ whose cohomology ring does not satisfy the obstruction, thus

$$M^n \not\hookrightarrow W^{n+1}.$$ 

The construction of $M$ has three steps.

- A good (bad?) multilinear function $F$ on $V$,
  (with $V = H^p(M)$, $F = \cup$ in mind),
- A commutative graded algebra $A = \bigoplus_{i=0}^{\infty} A_i$,
  (with $A = H^*(M)$ in mind),
- A smooth manifold $M$. 
Step 1: Special multilinear function

Recall that the cup product induces a symmetric/skew-symmetric multilinear function on $H^p(M)$.

**Note**

$(\wedge^q \mathbb{R}^m)^*$ denotes the space of all the $q$-th skew-symmetric multilinear functions on $\mathbb{R}^m$.

$(\vee^q \mathbb{R}^m)^*$ denotes the space of all the $q$-th symmetric multilinear functions on $\mathbb{R}^m$.

**Definition**

Let $F$ be an element of $(\wedge^q \mathbb{R}^m)^*$ (resp. $(\vee^q \mathbb{R}^m)^*$). We say that $F$ is **special** if there exist a subspace $U$ of $\mathbb{R}^m$ with $\dim U \geq \frac{m}{3}$ and a linear map $\varphi : U \to \mathbb{R}^m$ with no fixed non-zero vectors such that

for all $x_1, \ldots, x_q \in U$, $F(x_1, \ldots, x_q) = F(\varphi(x_1), \ldots, \varphi(x_q))$.

$\frac{m}{3}$ is related to $\frac{1}{2}(\beta_p(M) - \beta_{p+1}(W))$. 
Step 1: Special multilinear function (Cont.)

Definition

$F$ is called *special* if there exist a subspace $U$ of $\mathbb{R}^m$ with $\dim U \geq \frac{m}{3}$ and a linear map $\varphi : U \to \mathbb{R}^m$ with no fixed non-zero vectors such that for all $x_1, \ldots, x_q \in U$, $F(x_1, \ldots, x_q) = F(\varphi(x_1), \ldots, \varphi(x_q))$.

Proposition

Suppose $q \geq 3$ is an odd integer. If $m$ is sufficiently large, then there exists a proper closed subset $X_m$ of $(\wedge^q \mathbb{R}^m)^*$ (resp. $(\vee^q \mathbb{R}^m)^*$) such that all the special functions are in $X_m$.

- The condition “$q$ is odd” is crucial. If $q$ is even, then each $F$ is special. We may simply take $U = \mathbb{R}^m$ and $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ to be the map sending $x$ to $-x$ for all $x \in \mathbb{R}^m$.
- The conditions “$q \geq 3$” is necessary in the estimation.
**Step 2: From multilinear function to cohomology ring**

By using symmetric/skew-symmetric tensor product and dual space, we can prove

**Proposition**

Suppose that $p$ is a positive integer, $q \geq 3$ is an odd integer. Let $F \in (\bigwedge^q V)^*$ if $p$ is odd and $F \in (\bigvee^q V)^*$ if $p$ even, where $V$ is a finite dimensional vector space over $\mathbb{Q}$.

There exists a commutative graded algebra $A = \bigoplus_{i=0}^{pq} A_i$ satisfying Poincaré duality such that

(i) $A_p = V$, $A_{pq} = \mathbb{Q}$, and $A_i \neq 0$ only if $i$ is a multiple of $p$,
(ii) for all $v_1, \ldots, v_q \in A_p = V$, $F(v_1, \ldots, v_q) = v_1 \cup \ldots \cup v_q$.

**Note**

Commutativity means for all $a \in A_r$, $b \in A_s$, $a \cup b = (-1)^{rs}(b \cup a)$.

Poincaré duality means $A_n = \mathbb{Q}$ and $\varphi_i : A_i \to \text{Hom}(A_{n-i}, A_n)$ given by $\varphi_i(u)(v) = u \cup v$, $\forall u \in A_i$, $v \in A_{n-i}$ is an isomorphism for all $i$. 
Step 3: From cohomology ring to manifold

**Theorem (Sullivan 1977)**

For any commutative graded algebra $A = \bigoplus_{i=0}^{n} A_i$ over $\mathbb{Q}$, if $A$ satisfies Poincaré duality, $A_1 = 0$ and $A_{\frac{n}{2}} = 0$, then there exists a simply-connected, closed, smooth $n$-manifold $M$ such that $H^*(M; \mathbb{Q})$ is isomorphic to $A$.

Recall that $n = pq$, where $p \geq 2$, $q \geq 3$ and is odd.

- For $m \geq 3\beta_{p+1}(W)$, we can find a rational non-special multilinear function $F$.
- $F$ can be extended to a C.G.A. $A = \bigoplus_{i=0}^{q} A_{pi}$ over $\mathbb{Q}$.
- $p \geq 2 \implies A_1 = 0$,
  $q$ is odd $\implies A_{\frac{n}{2}} = 0$.

So we can finally get the desired manifold $M$. 
Positive result in dimension 4

Note
All manifolds are considered to be topological.

Theorem (a)
All simply-connected, indefinite, closed 4-manifolds can be embedded into an oriented closed 5-manifold.

Theorem (b)
All simply-connected, compact 4-manifolds with non-empty boundary can be embedded into $S^2 \tilde{\times} S^3$, the non-trivial $S^3$-bundle over $S^2$. 
Intersection forms on 4-manifolds

For any compact, connected, oriented 4-manifold $M$, the cup product

$$\cup : H^2(M, \partial M) \times H^2(M, \partial M) \to H^4(M, \partial M)$$

gives a symmetric bilinear form

$$Q_M : H_2(M) \times H_2(M) \to \mathbb{Z}$$

through duality theorem. Clearly, $Q_M(a, b) = 0$ if $a$ or $b$ is a torsion element. So $Q_M$ descends to an integral symmetric bilinear form on $H_2(M)/\text{Torsion} \cong \mathbb{Z}'$.

By choosing a basis of $\mathbb{Z}'$, $Q_M$ can be represented by a symmetric matrix $Q$. Poincaré theorem implies $\det Q = \pm 1$ when $M$ is closed and we say $Q_M$ is called unimodular.
Given an integral symmetric bilinear form $Q$ on $\mathbb{Z}^r$.

- $r$ is called the **rank** of $Q$, denoted by $\text{rk}(Q)$.
- Extend and diagonalize $Q$ over $\mathbb{Q}^r$, the number of positive entries and the number of negative entries are denoted by $b_2^+$ and $b_2^-$ respectively, the difference $b_2^+ - b_2^-$ is called the **signature** of $Q$, denoted by $\sigma(Q)$.
- $Q$ is called **indefinite** if both $b_2^+$ and $b_2^-$ are positive, and **definite** otherwise.
- $Q$ is called **even** if $Q(a, a)$ is even for any $a$, and **odd** otherwise.
Classification results

Given an integral unimodular symmetric bilinear form $Q$,

- If $Q$ is odd, then
  \[ Q \cong b_2^+ [+1] \oplus b_2^- [-1]. \]

- If $Q$ is even, then
  \[ Q \cong c_1 E_8 \oplus c_2 H, \]
  where $c_1, c_2 \in \mathbb{Z}$, $c_2 \geq 0$,
  
  $E_8$ is an even form with $\text{rk}(E_8) = \sigma(E_8) = 8$, and $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Theorem (Freedman)

Up to homeomorphism, there exists exactly one ($Q$ is even) or two ($Q$ is odd) simply-connected, closed, topological 4-manifold $M$ such that its intersection form is $Q$. 
Based on the following facts,

- Every indefinite form is build from $[+1]$, $[-1]$, $E_8$, $-E_8$ and $H$ by $\oplus$.
- $Q_M \oplus Q_{M'}$ is the intersection form of $M \# M'$.

we have

**Proposition**

There exist oriented closed connected 4-manifolds $M_i$, $1 \leq i \leq 7$, such that any simply-connected indefinite closed 4-manifold $M$ is homeomorphic to

$$\# k_1 M_1 \# k_2 M_2 \ldots \# k_7 M_7$$

for some non-negative integers $k_i$.

$M_1 = \mathbb{CP}^2$, $\ldots$, $M_3 = \overline{\mathbb{CP}}^2$, $\ldots$, $M_7 = S^2 \times S^2$. 
Sketch of the proof of theorem (a)

With direct construction, we have

**Lemma**

*For any oriented closed connected 4-manifold \( M \), there exists an oriented closed connected 5-manifold \( W \) such that for any positive integer \( r \), \( \#rM \) can be embedded into \( W \).*

For the each \( M_i \) in the above proposition, find corresponding \( W_i \) by this lemma, then choose

\[
W = W_1 \# W_2 \# \ldots \# W_7.
\]
Proof of the lemma

See figures.
Key points in the proof of theorem (b)

- The double of $M$, $DM = M\#(-M)$, is a simply-connected, indefinite, closed 4-manifold with $\sigma = 0$.

- $DM$ is homeomorphic to either $#k\mathbb{CP}^2#k\overline{\mathbb{CP}}^2$ or $#kS^2 \times S^2$.

- $DM$ is homeomorphic to either $#kS^2 \times S^2$ or $(k-1)S^2 \times S^2 \#S^2 \tilde{\times} S^2$.

- $#kS^2 \times S^2 \hookrightarrow S^5$, $S^2 \tilde{\times} S^2 \hookrightarrow S^2 \tilde{\times} S^3$.

Remark

Definite integral bilinear forms are far more complicated than indefinite ones.

Signature is not an obstruction for this embedding problem.
Thank you
ありがとう
감사합니다
谢谢