On embedding all n-manifolds into a single $(n+1)$-manifold

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Introduction

Question

Find the smallest nonnegative integer e_n, such that any n-dimensional connected, closed manifold can be embedded into a single connected, closed manifold of dimension $n + e_n$.

Note

All embeddings are considered to be topologically flat.

- $0 \leq e_n \leq n$, by Whitney embedding theorem,
- $e_0 = e_1 = 0$,
- $e_2 = 1$,
- $e_3 = 2$.

Why $e_2 = 1$?

Classification of connected, closed 2-manifolds

- Orientable surface: $\#nT^2$,
- non-orientable surface: $\#nT^2\#\mathbb{RP}^2$ or $\#nT^2\#\mathbb{RP}^2\#\mathbb{RP}^2$.

Lemma

If $M_1^n \hookrightarrow W_1^{n+1}$ and $M_2^n \hookrightarrow W_2^{n+1}$, then

$$M_1\#M_2 \hookrightarrow W_1\#W_2.$$

As $\#nT^2 \hookrightarrow S^3$, $\mathbb{RP}^2 \hookrightarrow \mathbb{RP}^3$, every surface embeds in

$$S^3\#\mathbb{RP}^3\#\mathbb{RP}^3 \cong \mathbb{RP}^3\#\mathbb{RP}^3.$$

Why $e_3 = 2$?

- Every oriented, closed 3-manifolds embeds into S^5. [Hirsch, 1961]
- Every non-orientable, closed 3-manifolds embeds into S^5. [Rohlin, 1965; Wall, 1965]
- There does not exist a single oriented, closed 4-manifold such that any connected, closed 3-manifold can be embedded into it. [Kawauchi, 1988]
- The condition "oriented" can be eliminated from the above statement. [Shiomi, 1991]

Does $e_n > 1$?

Question

Let n be a positive integer, whether there exists a connected, closed $(n+1)$-manifold W, such that any connected, closed n-manifold M can be embedded into W.

Up To Date Results

- YES for $n = 1, 2$,
- NO for $n = 3$ and $n = 4m - 1$, [Kawauchi].
- Partially YES for $n = 4$,
- No for n is a composite number and is not a power of 2.

We expect the answer to be NO for any $n \geq 4$.

Negative result

Theorem

If n is a composite number and is not a power of 2, then there does not exist a connected, closed $(n+1)$-manifold W, such that any smooth, simply-connected, closed n-manifold M can be embedded into W.

If such a W exists and is non-orientable, then its orientation double cover $\tilde{W}$ will also satisfies the condition, but $\tilde{W}$ is oriented.

Note

All manifolds are considered to be oriented, connected and closed.

Homological obstruction (in a simple case)

Suppose $M^n \hookrightarrow W^{n+1}$, then $W \backslash M$ could

- be connected
- have two components W_1 and W_2.

In the latter case, by using the Mayer-Vietoris sequence for (W_1, W_2), we get

Proposition

For any integer factorization $n = pq$, where $p, q > 0$, there exists a subspace $V \subset H^p(M; \mathbb{R})$ such that
(ii) $\dim V \geq \frac{1}{2}(\beta_p(M) - \beta_{p+1}(W))$, *and*
(iii) for any $x_1, \ldots, x_q \in V$, $x_1 \cup \ldots \cup x_q = 0$.

Homological obstruction (full version)

Proposition

Suppose $M^n \hookrightarrow W^{n+1}$, then for any integer factorization $n = pq$, where $p, q > 0$, there exists a subspace V of $H^p(M;\mathbb{R})$ and a linear transformation $\varphi : V \to H^p(M;\mathbb{R})$ such that
(i) φ has no fixed non-zero vectors,
(ii) $\dim V \geq \frac{1}{2}(\beta_p(M) - \beta_{p+1}(W))$, and
(iii) for any $x_1, \ldots, x_q \in V$, $x_1 \cup \ldots \cup x_q = \varphi(x_1) \cup \ldots \cup \varphi(x_q)$.

There are 3 items in this statement.

- a subspace
- a linear transformation
- a cup product relation

Sketch of the proof for the theorem

Let $n = pq$, where $p \geq 2$, $q \geq 3$ and is odd.
Given W^{n+1}, then $\beta_{p+1}(W)$ is fixed. We will find M^n whose cohomology ring does not satisfy the obstruction, thus

$$M^n \not\hookrightarrow W^{n+1}.$$

The construction of M has three steps.

- A good(bad?) multilinear function F on V, (with $V = H^p(M)$, $F = \cup$ in mind),
- A commutative graded algebra $A = \bigoplus_{i=0}^{\infty} A_i$, (with $A = H^*(M)$ in mind),
- A smooth manifold M.

Step 1: Special multilinear function

Recall that the cup product induces a symmetric/skew-symmetric multilinear function on $H^p(M)$.

Note

$(\wedge^q \mathbb{R}^m)^*$ denotes the space of all the q-th skew-symmetric multilinear functions on $\mathbb{R}^m$.
$(\vee^q \mathbb{R}^m)^*$ denotes the space of all the q-th symmetric multilinear functions on $\mathbb{R}^m$.

Definition

Let F be an element of $(\wedge^q \mathbb{R}^m)^*$ (resp. $(\vee^q \mathbb{R}^m)^*$). We say that F is *special* if there exist a subspace U of $\mathbb{R}^m$ with • $\dim U \geq \frac{m}{3}$ and • a linear map $\varphi : U \to \mathbb{R}^m$ with no fixed non-zero vectors such that

- for all $x_1, \ldots, x_q \in U$, $F(x_1, \ldots, x_q) = F(\varphi(x_1), \ldots, \varphi(x_q))$.

$\frac{m}{3}$ is related to $\frac{1}{2}(\beta_p(M) - \beta_{p+1}(W))$.

Step 1: Special multilinear function (Cont.)

Definition

F is called *special* if there exist a subspace U of $\mathbb{R}^m$ with $\dim U \geq \frac{m}{3}$ and a linear map $\varphi : U \to \mathbb{R}^m$ with no fixed non-zero vectors such that for all $x_1, \ldots, x_q \in U$, $F(x_1, \ldots, x_q) = F(\varphi(x_1), \ldots, \varphi(x_q))$.

Proposition

Suppose $q \geq 3$ is an odd integer. If m is sufficiently large, then there exists a proper closed subset X_m of $(\wedge^q \mathbb{R}^m)^*$ (resp. $(\vee^q \mathbb{R}^m)^*$) such that all the special functions are in X_m.

- The condition "q is odd" is crucial. If q is even, then each F is special. We may simply take $U = \mathbb{R}^m$ and $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ to be the map sending x to $-x$ for all $x \in \mathbb{R}^m$.
- The conditions "$q \geq 3$" is necessary in the estimation.

Step 2: From multilinear function to cohomology ring

By using symmetric/skew-symmetric tensor product and dual space, we can prove

Proposition

Suppose that p is a positive integer, $q \geq 3$ is an odd integer. Let $F \in (\wedge^q V)^*$ if p is odd and $F \in (\vee^q V)^*$ if p even, where V is a finite dimensional vector space over $\mathbb{Q}$.

There exists a commutative graded algebra $A = \bigoplus_{i=0}^{pq} A_i$ satisfying Poincaré duality such that
(i) $A_p = V$, $A_{pq} = \mathbb{Q}$, and $A_i \neq 0$ only if i is a multiple of p,
(ii) for all $v_1, \ldots, v_q \in A_p = V$, $F(v_1, \ldots, v_q) = v_1 \cup \ldots \cup v_q$.

Note

Commutativity means for all $a \in A_r, b \in A_s$, $a \cup b = (-1)^{rs}(b \cup a)$.
Poincaré duality means $A_n = \mathbb{Q}$ and $\varphi_i : A_i \to \mathrm{Hom}(A_{n-i}, A_n)$ given by $\varphi_i(u)(v) = u \cup v,\ \forall u \in A_i,\ v \in A_{n-i}$ is an isomorphism for all i.

Step 3: From cohomology ring to manifold

Theorem (Sullivan 1977)

For any commutative graded algebra $A = \bigoplus_{i=0}^{n} A_i$ over $\mathbb{Q}$,
If A satisfies Poincaré duality, $A_1 = 0$ and $A_{\frac{n}{2}} = 0$,
then there exists a simply-connected, closed, smooth n-manifold M such that $H^*(M;\mathbb{Q})$ is isomorphic to A.

Recall that $n = pq$, where $p \geq 2$, $q \geq 3$ and is odd.

- For $m \geq 3\beta_{p+1}(W)$, we can find a rational non-special multilinear function F.
- F can be extended to a C.G.A. $A = \bigoplus_{i=0}^{q} A_{pi}$ over $\mathbb{Q}$.
- $p \geq 2 \Longrightarrow A_1 = 0$,
  q is odd $\Longrightarrow A_{\frac{n}{2}} = 0$.

So we can finally get the desired manifold M.

Positive result in dimension 4

Note

All manifolds are considered to be topological.

Theorem (a)

All simply-connected, indefinite, closed 4-manifolds can be embedded into an oriented closed 5-manifold.

Theorem (b)

All simply-connected, compact 4-manifolds with non-empty boundary can be embedded into $S^2\tilde{\times}S^3$, the non-trivial S^3-bundle over S^2.

Intersection forms on 4-manifolds

For any compact, connected, oriented 4-manifold M, the cup product

$$\cup : H^2(M, \partial M) \times H^2(M, \partial M) \to H^4(M, \partial M)$$

gives a symmetric bilinear form

$$Q_M : H_2(M) \times H_2(M) \to \mathbb{Z}$$

through duality theorem. Clearly, $Q_M(a, b) = 0$ if a or b is a torsion element. So Q_M descends to an integral symmetric bilinear form on $H_2(M)/\mathrm{Torsion} \cong \mathbb{Z}^r$.

By choosing a basis of $\mathbb{Z}^r$, Q_M can be represented by a symmetric matrix Q. Poincaré theorem implies $\det Q = \pm 1$ when M is closed and we say Q_M is called unimodular.

Integral symmetric bilinear form

Given an integral symmetric bilinear form Q on $\mathbb{Z}^r$.

- r is called the rank of Q, denoted by $\mathrm{rk}(Q)$.
- Extend and diagonalize Q over $\mathbb{Q}^r$, the number of positive entries and the number of negative entries are denoted by b_2^+ and b_2^- respectively, the difference $b_2^+ - b_2^-$ is called the signature of Q, denoted by $\sigma(Q)$.
- Q is called indefinite if both b_2^+ and b_2^- are positive, and definite otherwise.
- Q is called even if $Q(a, a)$ is even for any a, and odd otherwise.

Classification results

Given an integral unimodular symmetric bilinear form Q,

- If Q is odd, then
$$Q \cong b_2^+[+1] \oplus b_2^-[-1].$$
- If Q is even, then
$$Q \cong c_1 E_8 \oplus c_2 H,$$
where $c_1, c_2 \in \mathbb{Z}$, $c_2 \geq 0$,
E_8 is an even form with $\mathrm{rk}(E_8) = \sigma(E_8) = 8$, and $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Theorem (Freedman)

Up to homeomorphism, there exists exactly one (Q is even) or two (Q is odd) simply-connected, closed, topological 4-manifold M such that its intersection form is Q.

Standard form of indefinite 4-manifolds

Based on the following facts,

- Every indefinite form is build from $[+1]$, $[-1]$, E_8, $-E_8$ and H by $\oplus$.
- $Q_M \oplus Q_{M'}$ is the intersection form of $M\#M'$.

we have

Proposition

There exist oriented closed connected 4-manifolds $M_i, 1 \leq i \leq 7$, such that any simply-connected indefinite closed 4-manifold M is homeomorphic to

$$\#k_1M_1\#k_2M_2\ldots\#k_7M_7$$

for some non-negative integers k_i.

$M_1 = \mathbb{CP}^2$,..., $M_3 = \overline{\mathbb{CP}}^2$,..., $M_7 = S^2 \times S^2$.

Sketch of the proof of theorem (a)

With direct construction, we have

Lemma

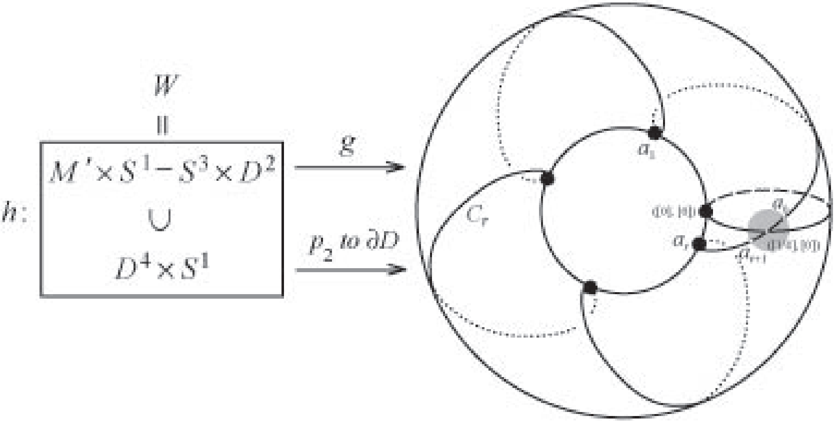
For any oriented closed connected 4-manifold M, there exists an oriented closed connected 5-manifold W such that for any positive integer r, $\#rM$ can be embedded into W.

For the each M_i in the above proposition, find corresponding W_i by this lemma, then choose

$$W = W_1 \# W_2 \# \ldots \# W_7.$$

Proof of the lemma

See figures.



Key points in the proof of theorem (b)

- The double of M, $DM = M\#(-M)$, is a simply-connected, indefinite, closed 4-manifold with $\sigma = 0$.
- DM is homeomorphic to either $\#k\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ or $\#kS^2 \times S^2$
- DM is homeomorphic to either $\#kS^2 \times S^2$ or $\#(k-1)S^2 \times S^2\#S^2\tilde{\times}S^2$.
- $\#kS^2 \times S^2 \hookrightarrow S^5$, $S^2\tilde{\times}S^2 \hookrightarrow S^2\tilde{\times}S^3$.

Remark

Definite integral bilinear forms are far more complicated than indefinite ones.

Signature is not an obstruction for this embedding problem.

Thank you

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