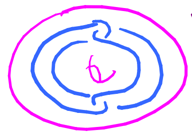
Slicing iterated Bing doubles

Jae Choon Cha
POSTECH

January 21, 2008

The 4^{th} East Asian School of Knots and Related Topics

Bing doubles

$BD :=$ 

V = standard $S^1 \times D^2$ in S^3

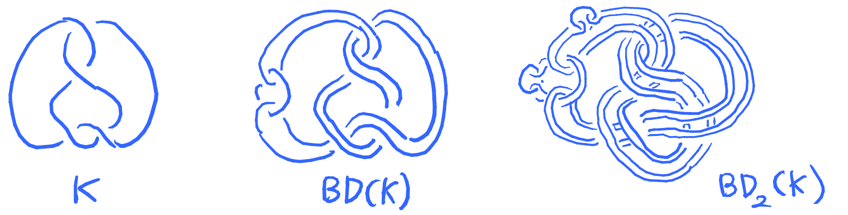
L = m-component link in S^3 $\Rightarrow$ Bing double $BD(L)$

$= (S^3 - \text{tub.nbhd. of } L) \underset{\partial}{\cup} \coprod^{m} (V, BD)$

along 0-framing

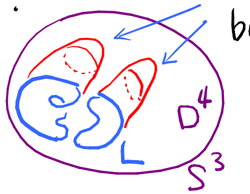
The n^{th} iterated Bing double $BD_n(L) := BD(BD_{n-1}(L))$

e.g.



Question : When is $BD_n(K)$ slice ?

A link $L \subset S^3$ is slice $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ disjoint 2-disks bounded by L

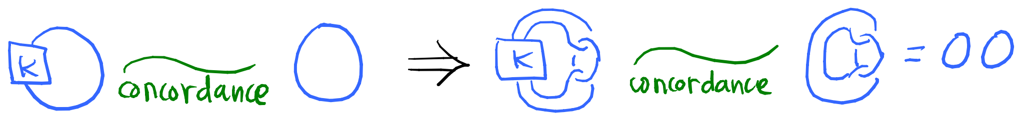


(locally flat or smooth)

Observation :

K is slice $\Rightarrow$ $BD_1(K)$ is slice $\Rightarrow \cdots \Rightarrow$ $BD_n(K)$ is slice $\Rightarrow \cdots$

(proof) Given a slicing disk $\Delta \subset D^4$ of K, there are disjoint 2-disks in a tub.nbhd $\Delta \times D^2$ bounded by $BD_1(K) \subset K \times D^2$:



Question : Is the converse true ?

$BD_n(K)$ slice $\overset{??}{\Leftarrow}$ $BD_{n+1}(K)$ slice

How to detect non-sliceness of $BD_n(K)$?

Many obstructions to being slice are useless for $BD_n(K)$ . . .

- linking number of $BD_n(K) = 0$
- Milnor's $\bar{\mu}$-invariants of $BD_n(K) = 0$

$\Leftarrow$ $BD_n(K)$ is a boundary link !

- Multivariable Alexander polynomial $\tilde{\Delta}_{BD_n(K)} = 1$

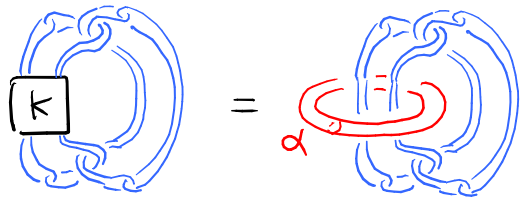
  [Kawauchi] L is slice $\Rightarrow$ $\tilde{\Delta}_L = f(t_1, \cdots)\, f(t_1^{-1}, \cdots)$

- Multivariable Levine-Tristram signature $\sigma_{BD_n(K)} = 0$

  [Cooper, $\cdots$] L is slice $\Rightarrow$ $\sigma_L = 0$

- Arf invariant of $BD_n(K) = 0$

Reason :



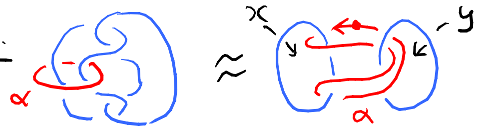
$\partial \cup$ (Exterior of K)

$BD_n(K)$ $\qquad$ $BD_n(O) = 2^n$-comp. unlink

$[\alpha] \in \pi_1(S^3 - BD_n(O)) =: F$ = free group on 2^n generators

is contained in $F^{(n)}$ = n^{th} derived series of F

e.g.



$\Rightarrow \alpha = xyx^{-1}y^{-1}$

$\Rightarrow$ Abelian invariants fail to detect (non-sliceness of) $BD_n(K)$ for $K > 0$

Recent new methods

[Teichner, Harvey, Cochran-Harvey-Leidy, 2006-7]

L^2-signature invariants associated to PTFA groups

[Cimasoni, 2006]

Sheiham's boundary link invariants

[C, 2006]

Hirzebruch-type invariants from iterated p-covers

[C-Livingston-Ruberman, C-T.Kim, 2006-7]

Covering link construction + $\left\{ \begin{array}{l} \text{abelian} \\ \text{Heegaard Floer} \\ L^2\text{-signature} \end{array} \right\}$ invariants

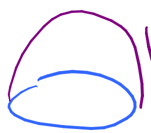
L^2-signatures and sliceness

M: closed 3-manifold

$\phi: \pi_1 M \to \Gamma$ (countable & discrete)

$\Rightarrow$ $\rho^{(2)}(M, \phi) \in \mathbb{R}$ is defined

Idea: For



W = slice disk exterior

M = surgery mfd of L

s.t.

$$\begin{array}{ccc} \pi_1 M & \xrightarrow{\phi} & \Gamma \\ i_* \downarrow & \nearrow \exists & \\ \pi_1 W & & \end{array},$$

sometimes one can show $\rho^{(2)}(M, \phi) = 0$ using W !

(L^2-index theory $\Rightarrow$ $\rho^{(2)}(M, \phi)$ can be computed from (the duality of) W)

So we need highly nontrivial $\pi_1 M \to \Gamma$ factoring through $\pi_1 W$.

Known useful $\pi_1 M \xrightarrow{\phi} \Gamma$ (factoring through $\pi_1 W$)

For knots : Cochran-Orr-Teichner (2003), Cochran-Teichner, Cochran-Kim $\Big\}$ highly solvable (non-abelian) coeff. gp. Γ

For links : Harvey (2006) Applied "injectivity theorem"

$$\underbrace{\pi_1 M / (\pi_1 M)^{(n)}_H}_{} \hookrightarrow \underbrace{\pi_1 W / (\pi_1 W)^{(n)}_H}_{}$$

torsion-free derived series

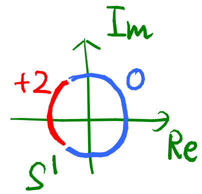
Applying L^2-invariants to $BD_n(K)$,

[Teichner (unpublished), Harvey '06]

$BD_n(K)$ is slice $\Rightarrow \int_{S^1} \sigma_K(\omega)\, d\omega = 0$

Levine-Tristram signature

e.g. $3_1 =$  : $BD_n(3_1)$ is not slice since $\sigma_K =$



Teichner, Cochran, Schneiderman, ··· asked :

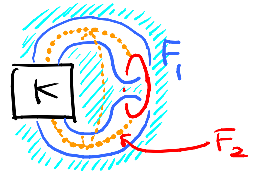
Conjecture For $4_1 =$  , $BD(4_1)$ is not slice.

Observation : $\sigma_K \equiv 0$ for $K = 4_1$

Since K is amphichiral

($\Rightarrow 2K = K \# K$ is slice
i.e, K is 2-torsion)

Boundary sliceness

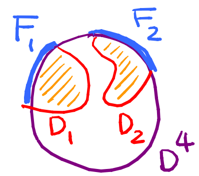


Recall $BD_n(K)$ is a boundary link,

i.e, components bound disjoint Seifert surfaces F_i

A boundary link L is boundary slice

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists$ slicing disks D_i s.t. the $F_i \cup D_i$ bound disjoint 3-mfds in D^4



[Sheiham '03] $\dfrac{\{\text{boundary links}\}}{\text{boundary slice (concordance)}} \overset{\exists}{\twoheadrightarrow} \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \oplus \mathbb{Z}_8^\infty$

[Cimasoni '06] $BD_1(K)$ boundary slice $\Longrightarrow$ K algebraically slice
(Seifert matrix $= \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}$)

Applying this, $BD_1(4_1)$ is not boundary slice.

A Hirzebruch-type invariant

Given $\left\{ \begin{array}{l} M : \text{closed 3-mfd} \\ \phi : \pi_1 M \to \Gamma \\ \mathbb{Z}\Gamma \to \mathbb{K} = \text{(skew) field} \\ \quad \text{with char} = 0 \end{array} \right\}$ such that $(M, \phi) = 0$ in $\Omega_3^{top}(B\Gamma)$,

i.e.,

Define $\lambda(M, \phi) := [\lambda_W^{\mathbb{K}}] - [\lambda_W^{\mathbb{Q}}] \in L^0(\mathbb{K}) =$ Witt group over $\mathbb{K}$

$\lambda_W^{\mathbb{K}}$ = $\mathbb{K}$-coefficient intersection form

$\lambda_W^{\mathbb{Q}}$ = ordinary intersection form

(take non-singular parts only!)

c.f. Signature defects due to Hirzebruch, Atiyah, Patodi, Singer, ···

[C, '06] $H_4(\Gamma; \mathbb{Z}) = 0 \Rightarrow \lambda(M, \phi)$ is independent of W.

Our main example : $\Gamma = \mathbb{Z}_d$, $\mathbb{Z}\Gamma \to \mathbb{K} = \mathbb{Q}(\zeta_d)$ ($\zeta_d = e^{2\pi\sqrt{-1}/d}$)

In this case, $L^0(\mathbb{Q}(\zeta_d))$ is not torsion free!

Invariants from iterated p-covers (p = prime)

$|\text{covering trans. gp.}| = p^r$

$M_n \to M_{n-1} \to \cdots \to M_0 = M$ tower of abelian p-covers

$\phi : \pi_1 M_n \to \mathbb{Z}_d$ ($d = p^a$) character

Given a p-structure $\mathcal{J} = (\{M_k\}, \phi)$, we consider $\lambda(M_n, \phi)$.
(when $(M_n, \phi) = 0$ in $\Omega_3^{top}(B\mathbb{Z}_d)$)

Remark :

① For any (non-abelian) p-cover $\tilde{M}$ of M, $\exists \{M_k\}$ s.t. $\tilde{M} = M_n$.

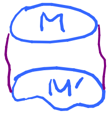
② M_n may be an irregular cover of M.

Alternative description : p-virtual character of $\pi_1(M)$

$\phi : H \to \mathbb{Z}_d$, $[\pi_1(M) : H] = p^a$ $\Longrightarrow$ $\exists$ a tower $\{M_i\}$ of iterated p-covers
($H \leq \pi_1(M)$)
s.t. $\pi_1(M_n) = H \leq \pi_1(M)$

$\rightsquigarrow \lambda(M_n, \phi)$

Theorem [C '06]

Suppose M and M' are $\mathbb{Z}_p$-homology cobordant, i.e. 

$\exists W$ s.t. $M \hookrightarrow W \hookleftarrow M'$ are $\mathbb{Z}_p$-homology equiv.

Then:

① $\{p\text{-structures of } M\} \approx \{p\text{-structures of } M'\}$

$\qquad (\{M_k\}, \phi) \qquad\qquad (\{M'_k\}, \phi')$

② $(M_n, \phi) = 0 \iff (M'_n, \phi') = 0$ in $\Omega_3^{top}(B\mathbb{Z}_d)$

In this case, $\lambda(M_n, \phi) = \lambda(M'_n, \phi')$ in $L^0(\mathbb{Q}(\zeta_d))$.

For a link L, we consider the surgery manifold M_L:

Corollary [C'06]

L is slice $\Rightarrow$ $\forall$ p-structure $(\{M_k\}, \phi)$ of M_L, $\lambda(M_n, \phi) = 0$

In fact, ① above is a consequence of the following:

$\mathbb{Z}$ localized at p
$\|$

For a group G, there defined $\widehat{G}$ = algebraic closure w.r.t. $\mathbb{Z}_{(p)}$-coeff.

= localization w.r.t. $\mathbb{Z}_{(p)}$-homology

[C,'06]

Thm $X \to Y$ induces $\widehat{\pi_1(X)} \cong \widehat{\pi_1(Y)}$

(e.g. it holds when $X \to Y$ is 2-connected i.e. $\cong$ on H_1, onto on H_2)

$\Rightarrow \{p\text{-structures of } Y\} \approx \{p\text{-structures of } X\}$

$\qquad (\{Y_k\}, \psi) \longmapsto (\{X_k\}, \phi)$ = pullback of $(\{Y_k\}, \psi)$

---

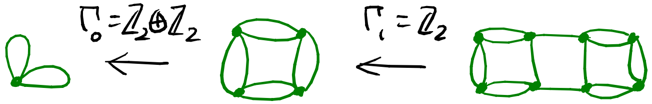
e.g. Boundary links with m components

Disjoint Seifert surfaces $\rightsquigarrow$ $M_L \xrightarrow{\text{2-connected}} X := \bigvee^m S^1$

Thom–Pontrygin construction

$\Rightarrow \{p\text{-structures of } M_L\} \approx \{p\text{-structures of } X\}$

easier to understand!

$m=2$: 

$X = X_0 \xleftarrow{\Gamma_0 = \mathbb{Z}_2 \oplus \mathbb{Z}_2} X_1 \xleftarrow{\Gamma_1 = \mathbb{Z}_2} X_2 \leftarrow \cdots$

In general, there are infinitely many such towers for $m > 1$:

Thm: $\{M_i\}$ = tower of p-covers of M_L with deck trans. gps $\{\Gamma_i\}$ $\Rightarrow$ $|\mathrm{Hom}(H_1 M_n, \mathbb{Z}_r)| = \mathbb{Z}_r^{(m-1)\prod_{i<n}|\Gamma_i| + 1}$

In many interesting cases,

(1) Our invariant $\lambda(M_n, \phi)$ can extract information from $\pi_1(M)^{(n)}$ for higher n

(2) It detects "torsion" (in an appropriate sense) as well as infinite order examples

$$L^0(\mathbb{Q}(\zeta_d)) \xrightarrow{\text{sign}} L^0(\mathbb{C}) \cong \mathbb{Z}$$

$$L^0(\mathbb{Q}(\zeta_d)) \xrightarrow{\text{dis}} \mathbb{Q}(\zeta_d + \bar\zeta_d)^\times / \{z\bar z \mid z \in \mathbb{Q}(\zeta_d)^\times\} \hookrightarrow \bigoplus_q \mathbb{Q}(\zeta^d + \bar\zeta^d)_q^\times / \{z\bar z \mid z \in \mathbb{Q}(\zeta^d)_q^\times\}$$

$$\cong \bigoplus_q \mathrm{Gal}\left[\mathbb{Q}(\zeta_d)_q / \mathbb{Q}(\zeta_d + \bar\zeta_d)_q\right] \cong \bigoplus_q \mathbb{Z}_2$$

dis $= (-1)^{r(r+1)/2} \det$

completion at q (Hasse principle)

norm residue symb. $(\mathrm{dis}\,\lambda(\cdots), D_q)_q \in$

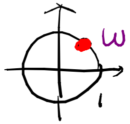
Artin Reciprocity

(3) $\lambda(M_n, \phi)$ is computed combinatorially.

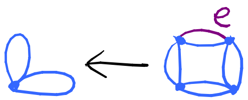
Theorem [C'06]

The Levine-Tristram signature σ_K is determined by $BD_n(K)$.
In particular, σ_K is nontrivial $\Rightarrow$ $BD_n(K)$ is not slice for all n.

Idea of proof: use $L^0(\mathbb{Q}(\zeta_d)) \xrightarrow{\text{sign}} \mathbb{Z}$

For any  $\omega = \exp(2\pi i s/p^a)$ with s, p^a arbitrary,

one can explicitly construct a p-structure $(\{M_i\}, \phi)$ of $M_{BD_n(K)}$ s.t. $\operatorname{sign} \lambda(M_n, \phi) = 2\sigma_K(\omega)$.

e.g. For $BD_1(K)$, 

$X_0 \qquad X_1$

and $\pi_1 X_1 \xrightarrow{\phi} \mathbb{Z}_{p^a}$, collapse all edges but e $\searrow$ $\pi_1 S^1 = \mathbb{Z}$ $\nearrow$ proj.

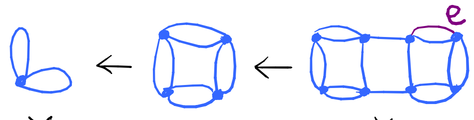
Theorem [C'06]

There are infinitely many (negatively) amphichiral knots K s.t. $BD_n(K)$ is not slice for all n. (including $K = 4_1$)

Idea of proof: use $L^0(\mathbb{Q}(\zeta_d)) \xrightarrow{\text{dis}} \dfrac{\mathbb{Q}(\zeta_d + \bar\zeta_d)^\times}{\{z\bar z \mid z \in \mathbb{Q}(\zeta_d)^\times\}} \xrightarrow{\text{norm residue symb.}} \mathbb{Z}_2$

One can explicitly construct a p-structure $(\{M_i\}, \phi)$ of $M_{BD_n(K)}$ s.t.

$$\operatorname{dis} \lambda(M_{n+1}, \phi) = \Delta_K(\sqrt[4]{-1})\,\Delta_K(-\sqrt[4]{-1})\,\Delta_K(-\sqrt{-1})\,\Delta_K(-1) \in \mathbb{Q}$$

e.g. For $BD_1(K)$, 

$X_0 \qquad X_1 \qquad X_2$

, $\pi_1 X_1 \xrightarrow{\phi} \mathbb{Z}_{p^a}$, $\searrow$ $\pi_1 S^1 = \mathbb{Z}$ $\nearrow$

More applications of the invariants from iterated p-covers:

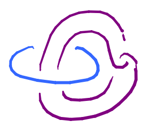
- "Exotic" homology cobordism classes of rational 3-spheres
  - Infinite family $\{\Sigma_i\}$ with distinct homology cob. types indistinguishable via previously known invariants
- Cochran-Orr-Teichner filtration $\{\mathcal{F}_{(h)}\}$ of link concordance classes
  - There are "torsion" elts in $\mathcal{F}_{(n)} - \mathcal{F}_{(n.5)}$ for all n.
- String link concordance "group" 
  - Our invariant gives homomorphisms into L-groups.
  - Kernel of Harvey's L^2-signature homomorphism is "large" (abelianization has ∞ rank, even modulo local knots)
- There are "independent" iterated Bing doubles (mod $\mathcal{F}_{(n.5)}$)

Covering links: a geometric method

$L \subseteq \Sigma = \mathbb{Z}_{(p)}$-homology 3-sphere $\rightsquigarrow$ $\tilde L \subseteq \tilde\Sigma$ = p-fold cyclic cover of Σ branched along K_0 (a $\mathbb{Z}_{(p)}$-homology sphere!)

$\tilde L$ = pre-image of L

K_0 = a component of L with $\mathbb{Q}/\mathbb{Z}$ self-linking $= 0$

e.g.  $\rightsquigarrow$ branching along ○ 

Def $\tilde L$ is called a p-covering link of L if

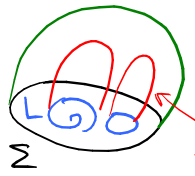
$$L = L_0 \to L_1 \to \cdots \to L_k = \tilde L$$

where L_{i+1} is $\begin{cases} \text{obtained from } L_i \text{ as above, or} \\ \text{a sublink of } L_i. \end{cases}$

For $R = \mathbb{Z}_{(p)}$ or $\mathbb{Q}$,

$L \subseteq \Sigma$ is called R-slice if

Σ = R-homology sphere



$\exists W$: R-homology 4-ball s.t. $\partial W = \Sigma$

$\exists$ disjoint 2-disks in W bounded by L

<u>Observation</u>: L is $\mathbb{Q}$-slice $\Leftrightarrow$ L is $\mathbb{Z}_{(p)}$-slice for some p.

<u>Proposition</u>: L is $\mathbb{Z}_{(p)}$-slice

$\Rightarrow$ any p-covering link $\tilde{L}$ of L is $\mathbb{Z}_{(p)}$-slice.

(proof) Take a branched cover of the $\mathbb{Z}_{(p)}$-homology 4-ball along a slice disk and check ...

Therefore, we need to investigate $\mathbb{Z}_{(p)}$ (or rational) concordance of knots and links!

<u>Rational ($\mathbb{Z}_{(p)}$-) concordance</u> : known results

[Cochran-Orr '93] Signature invariants of rational knot concordance

[C-Ko '02] Signature invariants from (rational) Seifert matrices of links in rational spheres

[C '07] Complete set of invariants of algebraic $\mathbb{Q}$-conc. gp. $\mathcal{G}_n$

Full computation of $\mathcal{G}_n$ $= \varinjlim$ (alg. conc. gp)

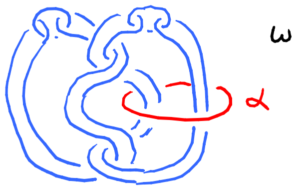
Geometric classification (n>1) c.f. $\mathbb{Q} = \varinjlim(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \cdots)$

Comparison with integral theory :

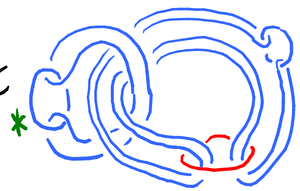
($\mathbb{Z}$-conc. gp) $\to$ ($\mathbb{Q}$-conc. gp) has highly large ker and coker.

L^2-signature invariants in dim=3 (n=1)

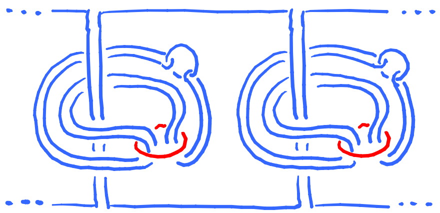
<u>Example</u> : covering link calculus simplifying $BD_2(K)$

$BD_2(K) =$ 

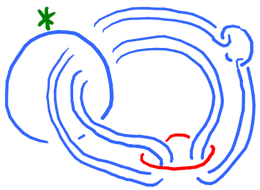
where K is tied along a 2-disk bounded by α

$\approx$ 

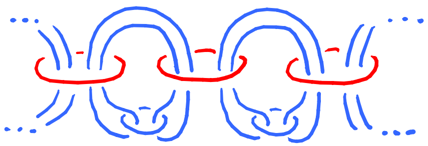
branching along *



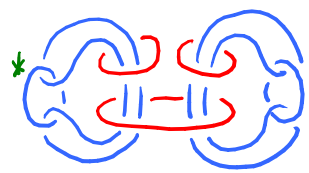
forgetting components



branching along *

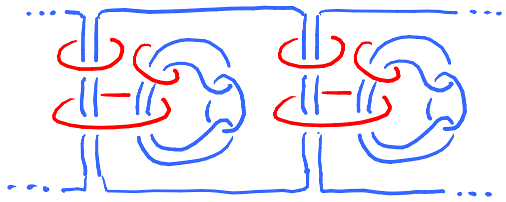


forgetting components

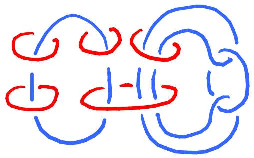


This is a p-covering link of $BD_2(K)$!

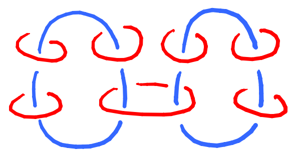
branching along *



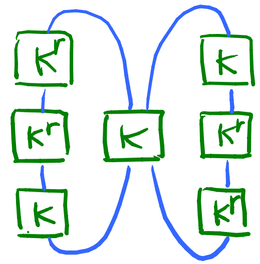
forgetting components



applying above arguments



=



From the above covering link argument, we have :

$BD_2(K)$ is $\mathbb{Z}_{(p)}$-slice $\Rightarrow 2K\#2K^r$ is $\mathbb{Z}_{(p)}$-slice

$\Rightarrow 4[K]=0$ in the algebraic $\mathbb{Z}_{(p)}$-concordance gp.

$\Rightarrow 4[K]=0$ in the algebraic concordance gp.

[Cochran-Orr '93], [C '07] : (alg. conc. gp) $\rightarrow$ (alg. $\mathbb{Z}_{(2)}$-conc. gp) is injective.

Combining such geometric argument with algebraic results of [Cochran-Orr '93], [C '07], [Kawauchi '80], we prove :

Theorem [C-Livingston-Ruberman '06]

$BD_1(K)$ is slice $\Rightarrow$ K is algebraically slice.

Recall : $BD_n(K)$ is slice $\Rightarrow BD_{n+1}(K)$ is slice --- (*)

Theorem [C-T.Kim '07]

For $n \geq 2$, the converse of (*) is rationally (or $\mathbb{Z}_{(p)}$-) true, i.e, for $R=\mathbb{Q}$ or $\mathbb{Z}_{(p)}$,

K is R-slice $\Rightarrow BD_1(K)$ is R-slice

$\Rightarrow BD_2(K)$ is R-slice $\Leftrightarrow BD_3(K)$ is R-slice $\Leftrightarrow \cdots$

Idea of proof :

For $n \geq 2$, $BD_{n+1}(K) \xrightarrow{\text{covering link construction}} BD_n(K)$

Theorem [C-T.Kim '07]

$BD_n(K)$ is slice for some $n \Rightarrow$ K is algebraically slice

Idea of proof : some further covering link calculus + algebraic results of [Kawauchi '80, Cochran-Orr '93, C '07]

Theorem [C-Livingston-Ruberman '06, C-T.Kim '07]

$BD_n(K)$ is slice for some n

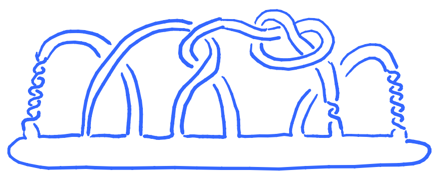
$\Rightarrow$ Ozsvath-Szabo invariant $\tau(K)=0$

Manolescu-Owens invariant $\delta(K)=0$

Idea of proof : $\left\{ \begin{array}{l} \text{Covering link calculus} \\ BD_n(K) \rightsquigarrow K\#K\#K^r\#K^r \end{array} \right\} + \left\{ \begin{array}{l} \tau(K\#J)=\tau(K)+\tau(J) \\ \tau(K^r)=\tau(K) \end{array} \right\}$

Theorem [C-T.Kim '07]

There are explicit examples of algebraically slice knots K s.t. $BD_n(K)$ is not slice for all n.

e.g. K = 

Idea of proof : Covering link calculus $\left(\begin{array}{c} BD_n(K) \\ \text{slice} \end{array} \Rightarrow \begin{array}{c} 2K\#2K^r \\ \mathbb{Q}\text{-slice} \end{array} \right)$

+ L^2-invariants of rational concordance [C '07, C-T.Kim '07]

c.f. [Cochran-Harvey-Leidy '07] proved the existence of such K.