Lattice Edge Number of Figure-8 knot

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Stick knots and Lattice Knots

- **Stick Knot**

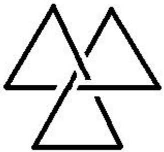
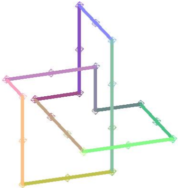
  A simple closed curve in $\mathbf{R}^3$ which consists of finite line segments.

- **Cubic Lattice**

  $\mathbf{Z}^3 = (\mathbf{R}^3 \times \mathbf{Z} \times \mathbf{Z}) \cup (\mathbf{Z} \times \mathbf{R}^3 \times \mathbf{Z}) \cup (\mathbf{Z} \times \mathbf{Z} \times \mathbf{R}^3)$

- **Lattice Knot**

  A stick knot in $\mathbf{Z}^3$

Definitions

For a knot type K,

- Stick Number $s(K)$

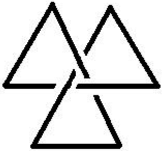
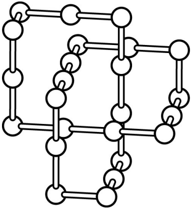
  minimum number of sticks required to construct a stick knot representation of K

- Latiitce Stick Number $s_L(K)$

  minimum number of sticks requireed to construct a lattice knot representation of K

- Lattice Edge Number $e_L(K)$

  minimum length of lattice knot representation of K

-  number of sticks $= 6$
-  number of sticks $= 6$, length $= 24$

Results

- [Huh-Oh 2005]
  $s_L(3_1) = 12$, $s_L(4_1) = 14$
  And $s_L(K) > 14$ for any other nontrivial knot K.

- [Diao 1993, 1994]
  $e_L(3_1) = 24$
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  $e_L(3_1) = 24$
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- Our Result
  $e_L(4_1) = 30$

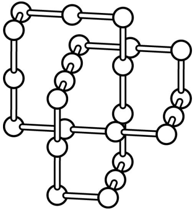
Presentation of Lattice Knot

- Vector Sequence

  For a given lattice knot, we can get a sequence of standard unit vectors **i**, **j**, **k**.

- Type of Lattice Knot

  A lattice knot is of the type l-m-n if the number of **i**(resp. **j**, **k**) is l(resp. m, n) in the vector sequence of the given knot.



i, **i**, **i**, **k**, **k**, −**j**, −**i**, −**i**, −**k**, −**k**, −**k**, **j**, **j**, **i**, **k**, **k**, −**j**, −**j**, −**j**, −**i**, −**i**, −**k**, **j**, **j**

Multiple Vector Sequence: 3**i**, 2**k**, −**j**, −2**i**, −3**k**, 2**j**, **i**, 2**k**, −3**j**, −2**i**, −**k**, 2**j**

4-4-4 type

Properly Leveled Cyclic Multiple Vector Sequence

Without lose of generality, every lattice knot can be written as a vector sequence $\{V_i\}_{i=1}^n$ such that $V_1 = \mathbf{i}$ and $V_n = \pm\mathbf{j}$ or $\pm\mathbf{k}$.

Definition

Let $\{V_i\}_{i=1}^n$ be a multiple vector sequence of a given lattice knot. Then $\{V_i\}_{i=1}^n$ is cyclic if

1. $n = 3m$
2. $V_{3k+1} = a\mathbf{i}$, $V_{3k+2} = b\mathbf{j}$, $V_{3k+3} = c\mathbf{k}$, a, b, $c \in \mathbf{Z}$

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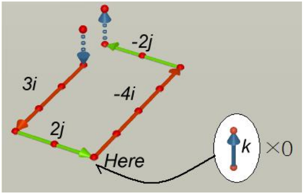

Properly Leveled

Let $\{V_i\}_{i=1}^{3n}$ be a cyclic multiple vector sequence.
Define x_k, y_k, and z_k by

- $x_k = \left(\sum_{l=0}^{3k+1} V_l\right) \cdot \mathbf{i}$, $y_k = \left(\sum_{l=0}^{3k+2} V_l\right) \cdot \mathbf{j}$, $z_k = \left(\sum_{l=0}^{3k+3} V_l\right) \cdot \mathbf{k}$

Definition

A cyclic multiple vector sequence $\{V_i\}_{i=1}^{3n}$ is properly leveled if

$$x_k \neq x_l,\ y_k \neq y_l,\ z_k \neq z_l, \text{ whenever } k \neq l$$

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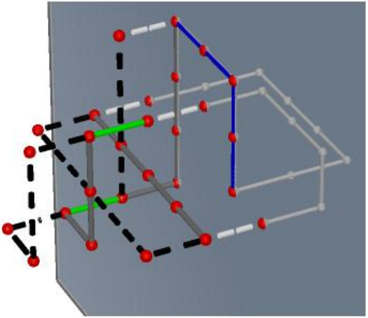


Lemma

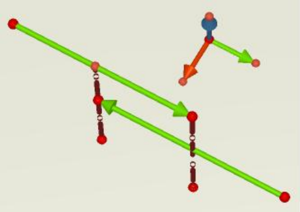
Let $\{V_i\}_{i=1}^{n}$ *be a multiple vector sequence of a lattice knot* K*. If* $\{V_i\}_{i=1}^{n}$ *is cyclic and properly leveled, then the projection of* K *onto* xy*-plane is regular.*

Lemma

Let $\{V_i\}_{i=1}^n$ be a multiple vector sequence of a lattice knot K. If $\{V_i\}_{i=1}^n$ is cyclic and properly leveled, then the projection of K onto xy-plane is regular.

Proof.

Suppose that $(3k+2)$-th stick and $(3l+2)$-th stick are overlap in xy-plane. Then $x_k = x_l$. It is impossible because $\{V_i\}_{i=1}^n$ is properly leveled. □



Tabulating of Lattice Knots

For each $n = 24, 26, 28, 30$, we had the following steps for the tabulation of lattice knot with length n.

1. Generate all vector sequence with length n as follows:
   1. Start with $\mathbf{i}$ and end with $\pm\mathbf{j}$ or $\pm\mathbf{k}$.
   2. $(\#\text{ of }\mathbf{i}) = (\#\text{ of }-\mathbf{i})$, and so on.
   3. Discard if there is a self intersection.
   4. Discard if the first appearance of y-axis direction vector is $-\mathbf{j}$, and so on.
   5. Discard if the length can be reduced.

2. Rewrite vector sequence as multiple vector sequence.
3. Add virtual edge($0\mathbf{i}$, $0\mathbf{j}$, $0\mathbf{k}$) to make cyclic.
4. Add new edges to make properly leveled.
5. Make Dowker-Thistlethwait notation from properly leveled cyclic multiple vector sequence.
6. Discard repeated Dowker-Thistlethwait notations
7. Identify knot using knotscape

Results

- $n = 24$

type	trivial knot	trefoil	total
4-4-4	5	1	6
5-4-3	1	0	1

- $n = 26$

type	trivial knot	trefoil	total
5-4-4	11	22	33
5-5-3	7	1	8
6-4-3	38	23	61

- $n = 28$

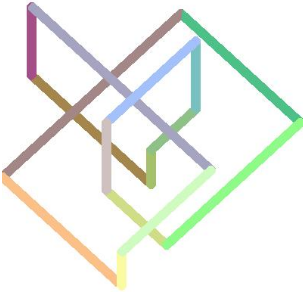
type	trivial knot	trefoil	total
5-5-4	118	92	210
6-4-4	126	132	258
6-5-3	101	79	180
7-4-3	33	22	55

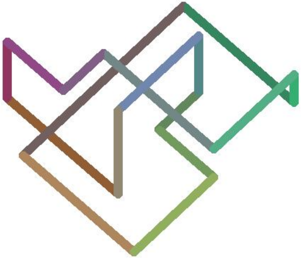
Lattice knot with length 30

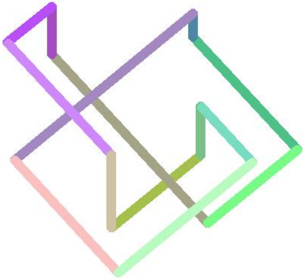
- type 8-4-3, 8-5-2, 7-4-4, 7-5-3, 7-6-2, 6-6-3, 6-5-4 are all trivial knots or trefoil knots
- type 6-5-4

vector sequence	DT-notation	
111133522444663352551166322244	6 -10 -8 2 -4	4_1
111133224255116664455323322644	-10 14 -12 4 -16 -6 -8 2	4_1
111133222466115554426633322445	4 6 8 2	4_1

1=**i**, 2=−**i**, 3=**j**, 4=−**j**, 5=**k**, 6=−**k**







Thank You!