Commensurability of Surface Automorphisms

Hongbin Sun

School of Mathematical Sciences
Peking University

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Outline

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- Two manifold automorphisms (M_1, ϕ_1), (M_2, ϕ_2) are said to be rational commensurable if there are $l_1, l_2 \in \mathbb{Z}_+$, such that $(M_1, \phi_1^{l_1})$, $(M_2, \phi_2^{l_2})$ are commensurable.

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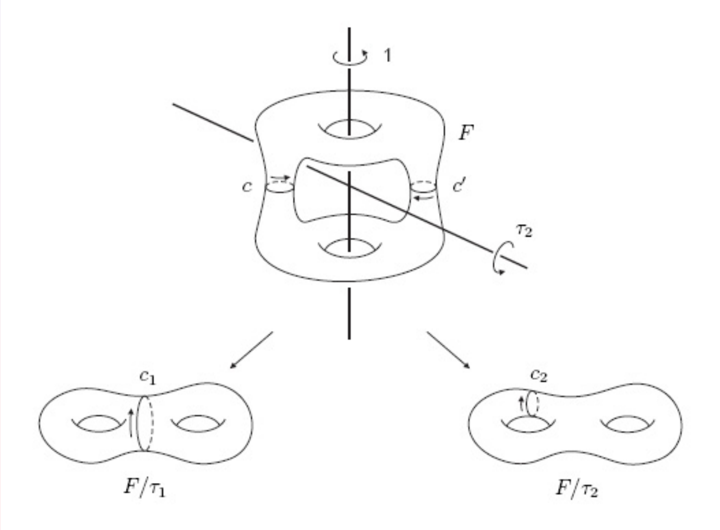
- All the surface automorphism ϕ we consider here satisfies: ϕ^k is generated by Dehn twist along disjoint essential circles for some $k \in \mathbb{Z}_+$;
- We call this type of surface automorphism pseudo D-type automorphism.

Example



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- $\Sigma(\phi) = \{\Sigma \mid \Sigma \text{ is a component of } F - N(\Gamma(\phi))\}$;
- $\forall\ \Sigma \in \Sigma(\phi),\ \Omega(\Sigma) = \{\gamma \mid \gamma \text{ is a component of } \partial\Sigma, \text{ but not a component of } \partial F\}$.

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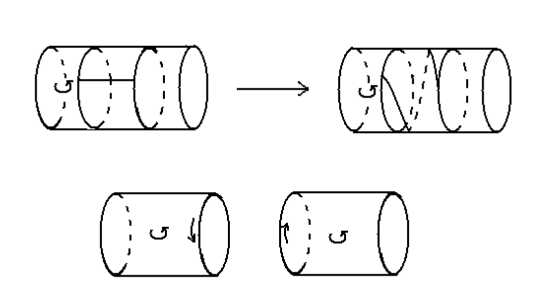
- 

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- $\lambda(\phi)_{(p,q)} = \frac{\sum_{\Sigma \in \Sigma(\phi)(p,q)} \chi(\Sigma)}{\chi(F)}$.

Definitions

- Polynomial pair: $p(\phi)(x,y) = (p_1(\phi)(x,y), p_2(\phi)(x,y))$
  $= \sum_{(p,q)\in\mathbb{Q}^2}(p,q)\lambda(\phi)_{(p,q)}x^p y^q;$

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  $= \sum_{(p,q)\in\mathbb{Q}^2}(p, q)\lambda(\phi)_{(p,q)}x^p y^q$;
- $p(x, y)$ is projectively equal to $q(x, y)$ ($p(x, y) \stackrel{p}{=} q(x, y)$)
  if $p_1(x, y) = q_1(x, y)$, $p_2(x, y) = q_2(x, y)$
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  or $p_1(x,y) = q_2(y,x)$, $p_2(x,y) = q_1(y,x)$;
- $A(\phi) = \frac{1}{2}p(\phi)(1,1)$;
- $(p_1, p_2) \stackrel{p}{=} (q_1, q_2)$
  if $(p_1, p_2) = (q_1, q_2)$ or $(p_1, p_2) = (q_2, q_1)$.

The Main Theorem

If two pseudo D-type automorphisms (F_1, ϕ_1), (F_2, ϕ_2) are commensurable, then

$$p(\phi_1)(x, y) \overset{p}{=} p(\phi_2)(x, y)$$

and $A(\phi_1) \overset{p}{=} A(\phi_2)$.

Corollary

If two pseudo D-type automorphisms (F_1, ϕ_1), (F_2, ϕ_2) are rational commensurable, then there is a rational number $s \in \mathbb{Q}_+$ such that:

$$p(\phi_1)(x, y) \overset{p}{=} s \times p(\phi_2)(x, y)$$

and $A(\phi_1) \overset{p}{=} s \times A(\phi_2)$.

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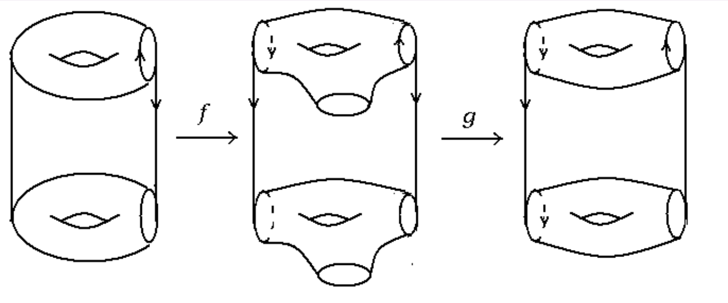
- In fact, we can construct infinite different structures of surface bundle over circle up to rational commensurability on some 3 manifold M.

Example 1

- *M*:



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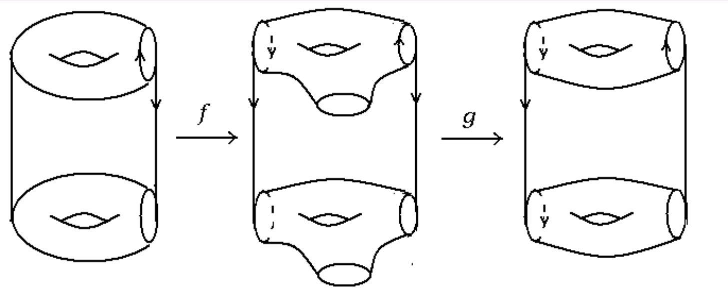

- $f(1,0) = (1,0) \quad f(0,1) = (-1,1);$
  $g(1,0) = (1,0) \quad g(0,1) = (1,1).$

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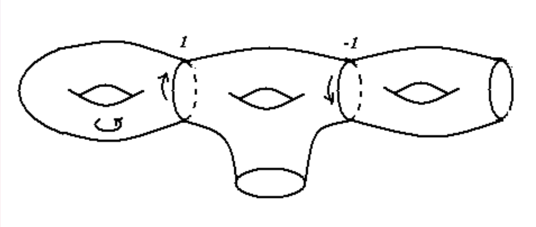
- (F_1, ϕ_1)



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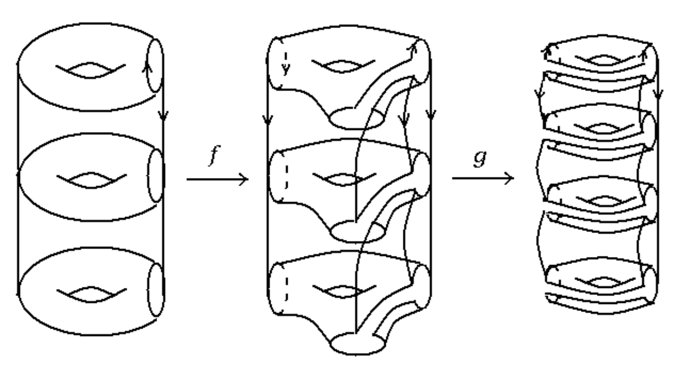
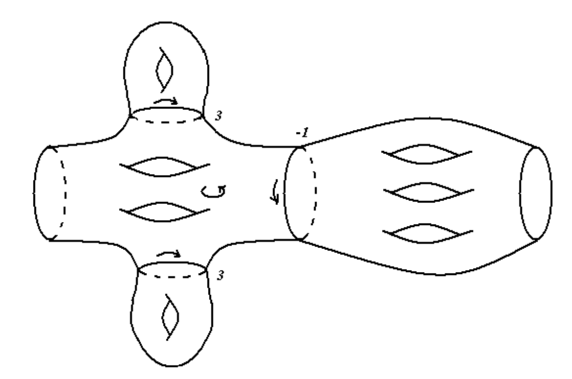
- Another version of M:



Example 1

- (F_2, ϕ_2^6)



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- For any $i \neq j \in \mathbb{Z}_+$, (F_i, ϕ_i), (F_j, ϕ_j) are not rational commensurable.

Example 2

We can also construct $M = F_1 \times I/\phi_1 = F_2 \times I/\phi_2$, $(F_1, \phi_1), (F_2, \phi_2)$ are not rational commensurable, and $g(F_1) = g(F_2)$.

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- Whether there are infinite (F_i, ϕ_i), such that $M = F_i \times I/\phi_i$, and there is an integer $g = g(F_i)$, $i = 1, 2, \ldots$ and for any $i \neq j \in \mathbb{Z}_+$, (F_i, ϕ_i) and (F_j, ϕ_j) are not rational commensurable;

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- If $M = F_1 \times I/\phi_1 = F_2 \times I/\phi_2$, then is there any explicit relation between (F_1, ϕ_1), (F_2, ϕ_2)?

Thank You !