

# Prime knots with arc index up to 11 and an upper bound of arc index for non-alternating knots

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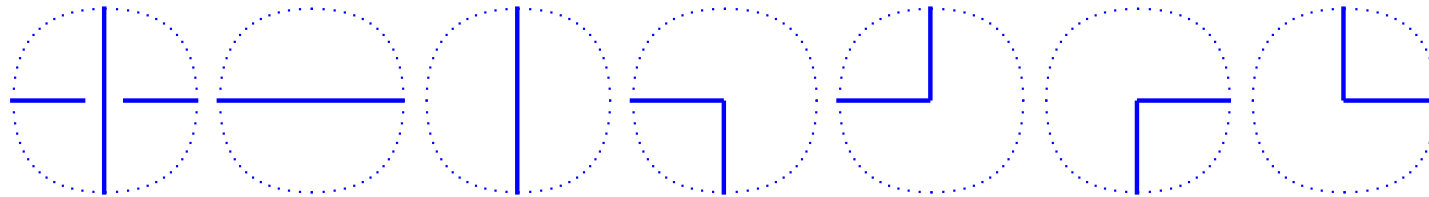
The Fourth East Asian School of Knots and Related Topics  
University of Tokyo

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- Arc presentation and arc index
- Tabulation of prime knots by arc index
- An upperbound of arc index for non-alternating knots

# Cromwell diagram

Cromwell showed that every link diagram is isotopic to a diagram which is a finite union of the following local diagrams in such a way that no more than two corners exist in any vertical line and any horizontal line.



Cromwell's local diagrams

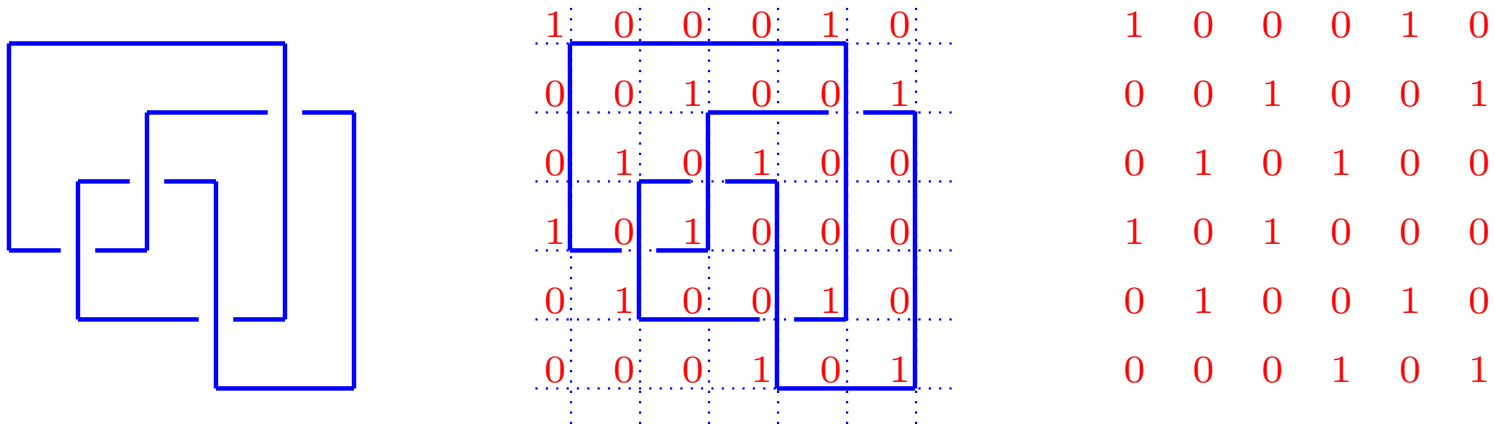
Such a diagram is called a *Cromwell diagram*.



Cromwell diagram of a trefoil knot

## Cromwell matrix

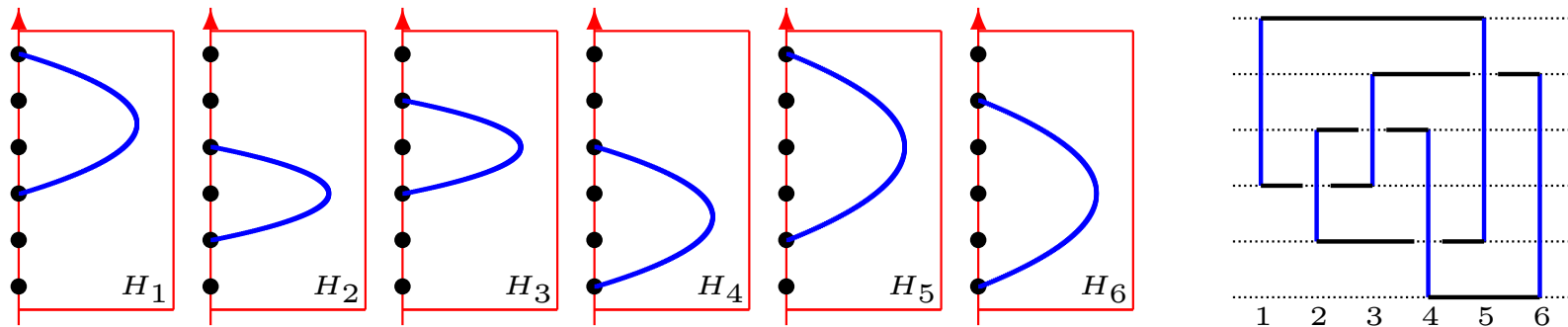
An  $n \times n$  matrix each of whose rows and columns has exactly two 1's and 0's elsewhere is called a *Cromwell matrix*. By joining two 1's in each column of a Cromwell matrix with a vertical line segment and two 1's in each row with a horizontal line segment which underpasses any vertical line segments that it crosses, we obtain its Cromwell diagram. Conversely, given a Cromwell diagram with  $n$  horizontal lines and  $n$  vertical lines, we place 1's at each corner and 0's at other points where the lines and their extensions cross, to construct its Cromwell matrix.



Construction of Cromwell matrix from Cromwell diagram and its inverse

# Arc presentation

An *arc presentation* of a knot or a link  $L$  is an ambient isotopic image of  $L$  contained in the union of finitely many half planes, called *pages*, with a common boundary line in such a way that each half plane contains a properly embedded single arc.

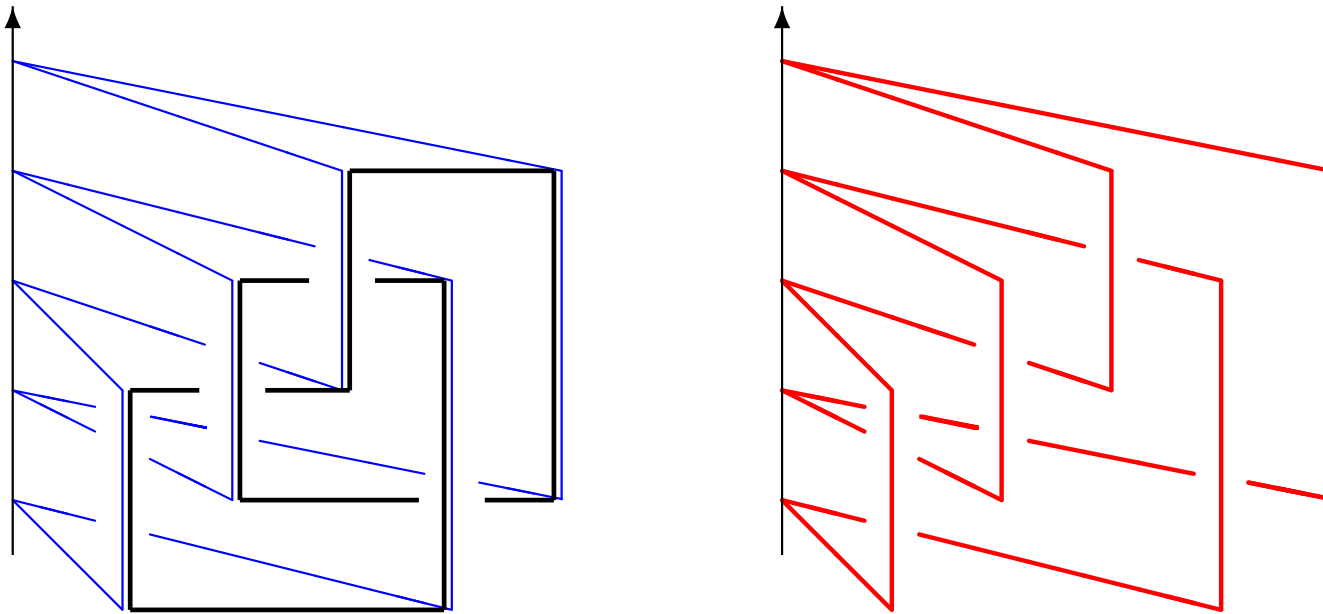


An arc presentation and its projection of the figure eight knot

**Proposition 1 (Cromwell 1995)** *Every link admits an arc presentation.*

## Proof of Proposition 1 (Existence of arc presentation)

As illustrated below, the horizontal line segments of a Cromwell diagram are horizontally pulled backwards to touch a vertical axis behind the diagram to form an arc presentation. As every link has a Cromwell diagram, we can conclude that every link admits an arc presentation.



Arc presentation constructed from a Cromwell diagram

## Arc index

The minimal number of pages among all arc presentations of a link  $L$  is called the *arc index* of  $L$  and is denoted by  $\alpha(L)$ .

$\alpha(L)$	2	3	4	5
$L$	unknot	none	2-component unlink, Hopf link	trefoil

Links with arc index up to 5

- Nutt identified all knots up to arc index 9. (1999, Math. Proc. Camb. Phil. Soc. **126**)
- Beltrami determined arc index for prime knots up to 10 crossings. (2002, JKTR **11**(3) — Proc. Knots 2000 Korea v.1)
- J-Kim-Lee-Gong-Kim-Kim-Oh identified all prime knots up to arc index 10. (2007, Knots and Everything **40** — Proc. ILDT 2006)
- Ng determined arc index for prime knots up to eleven crossings. (2006, arXiv:math/0612356)

## Properties of arc index

**Theorem 2 (1995, Cromwell)**  $\alpha(K_1 \sharp K_2) = \alpha(K_1) + \alpha(K_2) - 2$ .

**Theorem 3 (2000, Bae-Park)** *If  $L$  is a non-split link then*

$$\alpha(L) \leq c(L) + 2$$

*where  $c(L)$  is the minimal crossing number of  $L$ . The equality holds if and only if  $L$  is alternating.*

**Lemma 4** *There are finitely many knots with arc index  $n$  for each  $n \geq 2$ .*



## Allowable moves on Cromwell matrices

Any finite combination of the following moves on a Cromwell matrix does not change the link type of the corresponding Cromwell diagram up to mirror images.

**M1.** Flipping in a horizontal axis, a vertical axis, a diagonal axis or an anti-diagonal axis.

**M2.** Rotation in the plane by 90 degrees.

**M3.** Moving the first row to the bottom or the first column to the rear.

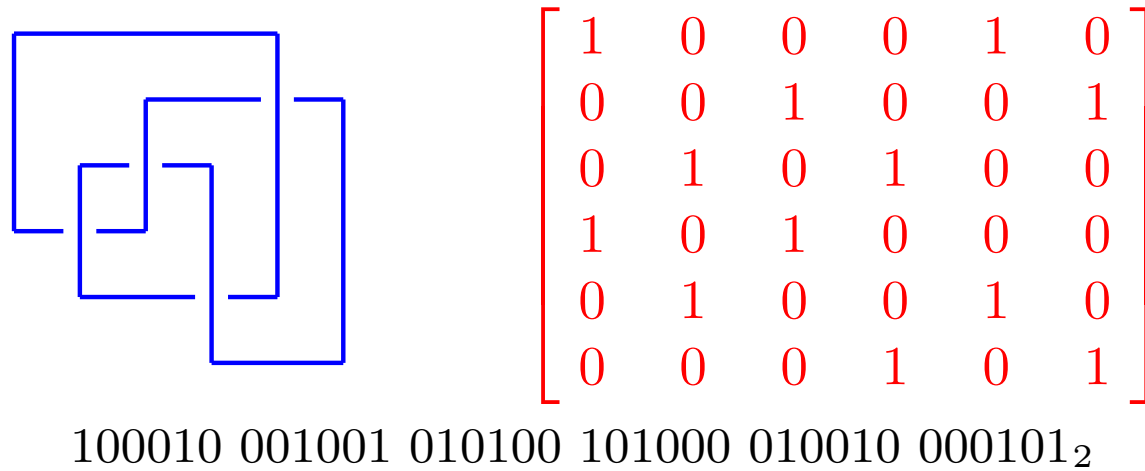
**M4.** Interchange of two adjacent rows or columns whose ones are in ‘non-interleaving’ position.

$$\begin{bmatrix} \dots & & & & & & \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 \\ \dots & & & & & & \end{bmatrix} \iff \begin{bmatrix} \dots & & & & & & \\ 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ \dots & & & & & & \end{bmatrix}$$

Interchange of adjacent rows in a non-interleaving position

## The norm of a Cromwell matrix

The *norm* of a Cromwell matrix is the  $n^2$  digit binary number obtained by concatenating its rows. An example is shown below.

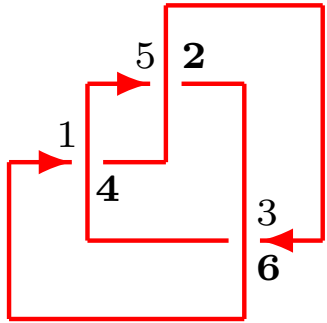


## Tabulation procedure

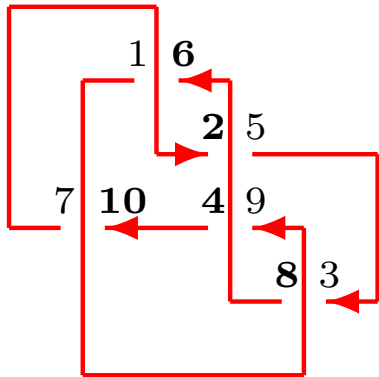
To tabulate prime knots with arc index up to 11, we proceeded with the following steps, for each integer  $n = 5, \dots, 11$ .

- (1) In the norm-decreasing order, generate all  $n \times n$  Cromwell matrices whose leading entry of the first row is 1.
- (2) Discard if it corresponds to a link of more than one components.
- (3) Discard if its Cromwell diagram is not prime.
- (4) Discard if a sequence of moves M1–M4 ever increase the norm.
- (5) Discard if a finite sequence of move M4 ever makes two 1's adjacent horizontally or vertically, as their existence causes a reduction of the size of Cromwell matrix.
- (6) Identify the knot of its Cromwell diagram.
- (7) Discard the knot if it already appeared for  $n$  or for  $k < n$ .

# Dowker-Thistlethwaite codes



1 3 5  
4 6 2      4 6 2



1 3 5 7 9      -6 8 -2 -10 4  
-6 8 -2 -10 4

## A summary of our tabulation

Our computer program in which the steps (1)–(5) were implemented, produced 663,341 Cromwell matrices for  $n = 11$  and their Dowker-Thistlethwaite codes (‘DT codes’ for short). Using ‘Knotscape’, we were able to eliminate most of the duplications and obtained 2,727 distinct DT codes of prime knots. They include

- All prime knots up to arc index 11 except the  $13n4639$  having arc index 10. There are 2,616 such knots.
- All prime knots up to 11 crossings except the alternating knots with 10 or 11 crossings. There are 311 such knots.
- One duplication among 13 crossing knots, three among 14 crossing knots and two among 15 crossing knots.
- 105 duplications among 392 knots with 17 crossings or higher.

These duplications were detected by using polynomial invariants and hyperbolic invariants computed by Knotscape.

# Prime knots up to arc index 11 or up to 12 crossings

Arc index	5	6	7	8	9	10	11	12	13	14	Total
Crossings											
3	1										1
4		1									1
5			2								2
6				3							3
7					7						7
8			1	2		18					21
9				2	6		41				49
10				1	9	32		123			165
11					4	46	135		367		552
12					2	48	211	627		1288	2176
13						49	399				
14						17	477				
15					1	22	441				
16						7	345				
17						1	191				
18							76				
19							12				
20							3				
21							3				
24							1				
Total	1	1	3	8	29	240	2335				

## Non-alternating prime links

The numbers on the main diagonal of the table in the previous page count alternating knots. The following is our main theorem.

**Theorem 5** *A prime link  $L$  is non-alternating if and only if*

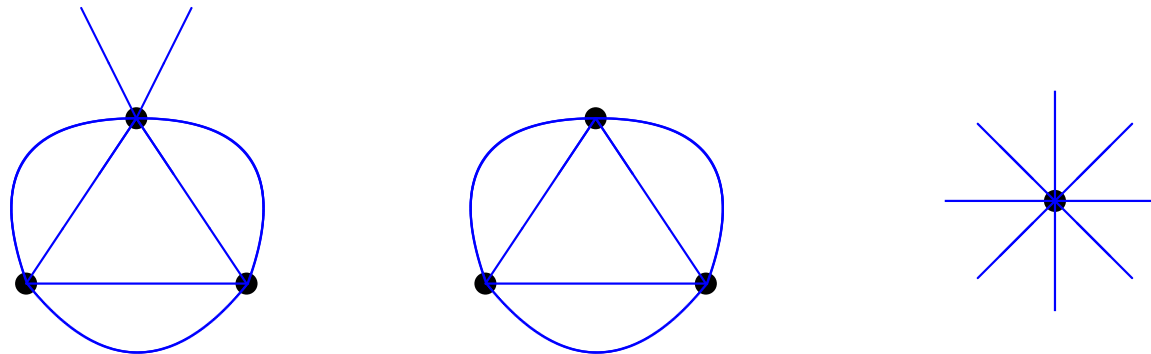
$$\alpha(L) \leq c(L).$$

- This inequality explains why the boxes one step below the main diagonal are all empty.
- Beltrami obtained an inequality sharper than this for semi-alternating links.
- This theorem allows us to put 627 in the table for the number of 12 crossing knots with arc index 12.

# Knot-spoke diagram

A *knot-spoke diagram*  $D$  is a finite connected plane graph satisfying

1. There are three kinds of vertices in  $D$ ; a *distinguished vertex*  $v_0$  with valency at least four, 4-valent vertices, and 1-valent vertices.
2. Every edge incident to a 1-valent vertex is also incident to  $v_0$ . Such an edge is called a *spoke*.



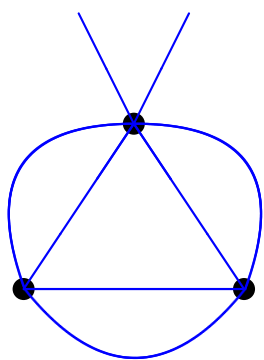
Knot-spoke diagrams

A *wheel diagram* is a knot-spoke diagram without any non-spoke edges.

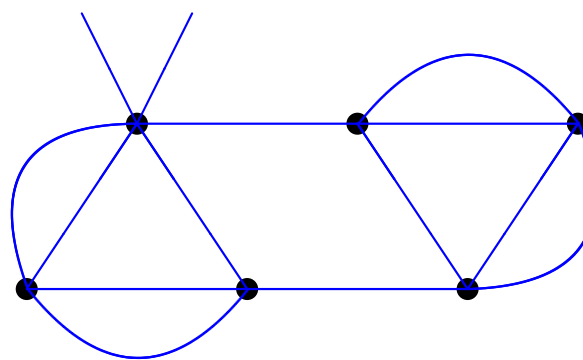


## Prime knot-spoke diagrams

A knot-spoke diagram  $D$  is said to be *prime* if no simple closed curve meeting  $D$  in two interior points of edges separates multi-valent vertices into two parts.



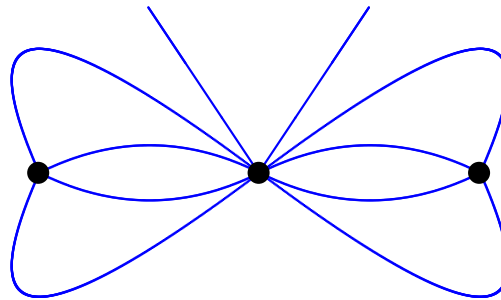
prime



non-prime

# Cut-point

A multi-valent vertex  $v$  of a knot-spoke diagram  $D$  is said to be a *cut-point* if there is a simple closed curve  $S$  meeting  $D$  in  $v$  and separating non-spoke edges into two parts.



Cut-point

- A cut-point free knot-spoke diagram with more than one non-spoke edges cannot have a *loop*.
- If a prime knot-spoke diagram  $D$  has a cut-point, then the distinguished vertex  $v_0$  must be the cut-point with valency bigger than four.

## Assigning relative edge-heights at vertices

To obtain the type of a knot or link which can be projected onto a knot-spoke diagram  $D$ , we may assign relative heights of the endpoints of edges of  $D$  in the following way.

1. At every 4-valent vertex, pairs of opposite edges meet in two distinct levels so that a knot-crossing is created.
2. If the distinguished vertex  $v_0$  is incident to  $2a$  non-spoke edges and  $b$  spokes, then its small neighborhood is the projection of  $n = a + b$  arcs at distinct levels whose relative heights can be specified by the numbers  $1, \dots, n$ . Every spoke is understood as the projection of an arc on a vertical plane whose endpoints project to  $v_0$ .

## Knot spoke diagram of a knot or a link

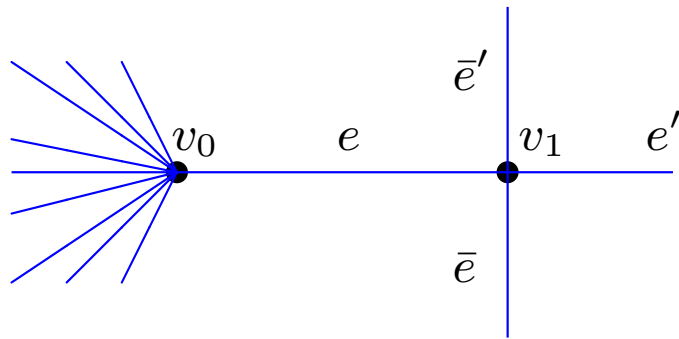
Suppose a knot-spoke diagram  $D$  has height information at multi-valent vertices, then

1.  $D$  determines a knot(or link)  $L$ . In this case, we say that  $D$  is a knot-spoke diagram of  $L$ .
2. If  $D$  has only 4-valent vertices then it is a knot(or link) diagram.
3. If  $D$  has no non-spoke edges, i.e., if  $D$  is a wheel diagram, then it is an arc presentation.

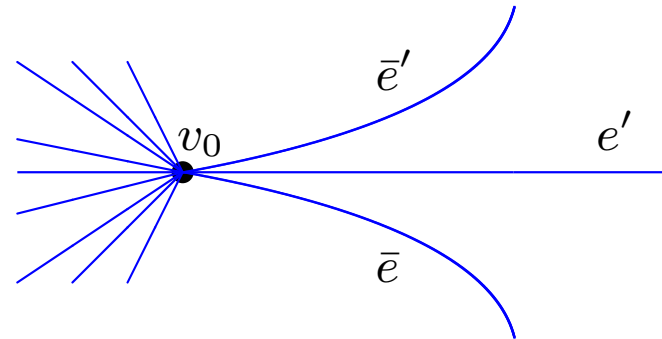
## Contracting an edge incident to $v_0$

Let  $e$  be an edge of a cut-point free knot-spoke diagram  $D$  as in the figure. The knot-spoke diagram  $D_e$  is obtained by

- contracting  $e$  and
- replacing any simple loop thus created by a spoke.



Local diagram of  $D$  near  $e$



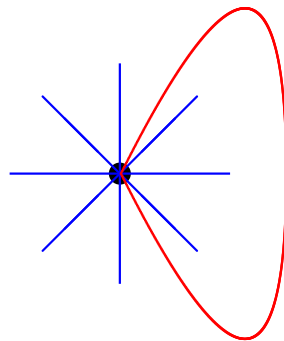
Local diagram of  $D_e$  near  $v_0$

A loop in a knot-spoke diagram is said to be *simple* if the other non-spoke edges are in one side of it.

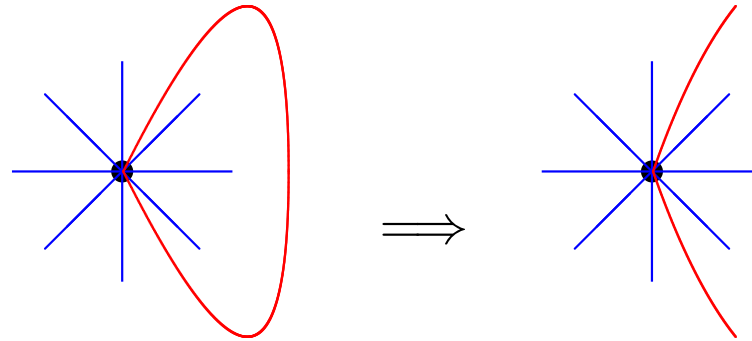
## $D$ and $D_e$

There are two important facts to point out.

1.  $D$  and  $D_e$  represent the same knot or link.
2. The sum of the number of regions divided by the non-spoke edges and the number of spokes is unchanged.
3.  $D_e$  is prime if  $D$  is prime.
4. Starting from a knot diagram  $D$ , we end up with a knot-spoke diagram with  $c(D)$  spokes and only one non-spoke edge which is a non-simple loop where  $c(D)$  is the number of crossings in  $D$ .



## Wheel diagram with $c(D) + 2$ spokes



The last non-spoke edge, which is a loop, is being folded to create two extra spokes. This shows the inequality

$$\alpha(L) \leq c(L) + 2$$

of Theorem 3.

## Basic tool

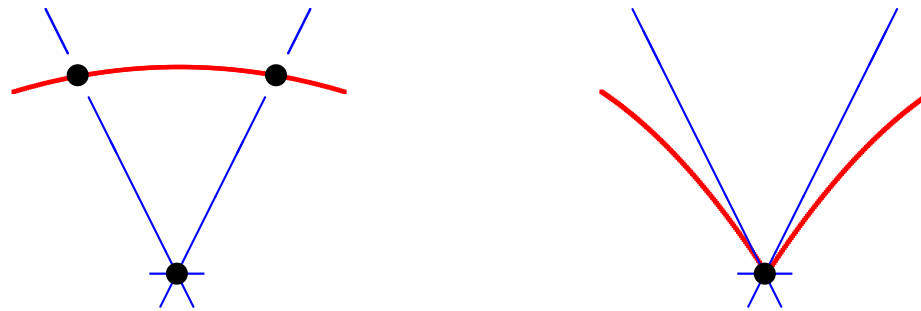
The condition ‘4’ above is guaranteed by the lemma below.

**Lemma 6 (Bae-Park)** *Let  $D$  be a knot-spoke diagram without cut-points. Suppose that  $D$  has at least two multi-valent vertices. Then there are at least two non-loop non-spoke edges  $e$  and  $f$ , incident to  $v_0$ , such that the knot-spoke diagrams  $D_e$  and  $D_f$  have no cut-points.*



## Idea of Proof of Theorem 5

Find a sequence of edge-contractions toward a wheel diagram so that there are at least two occasions that the number of regions is reduced by one without creating a spoke.



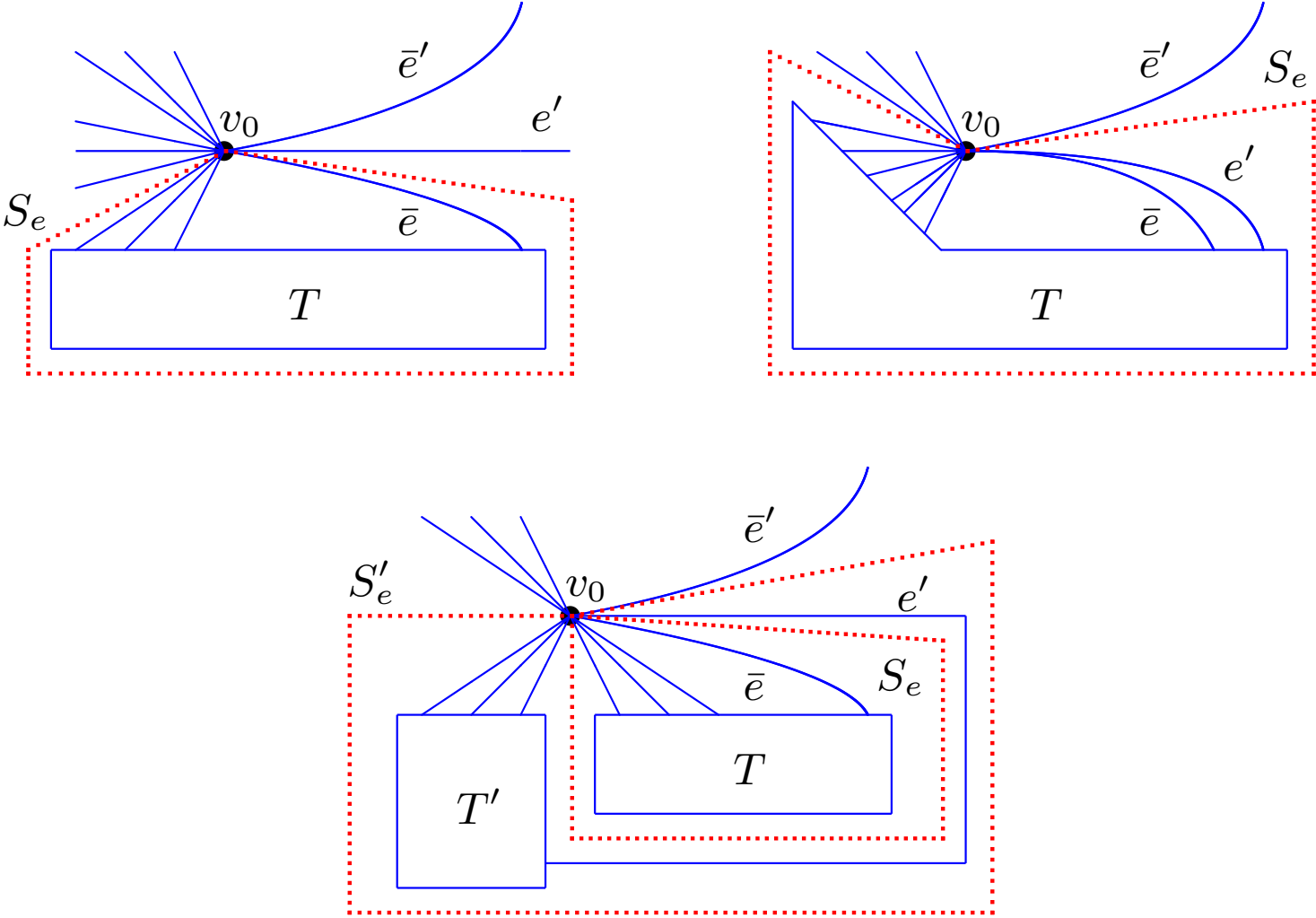
Reducing the number of regions by one without creating a spoke

## Main tool

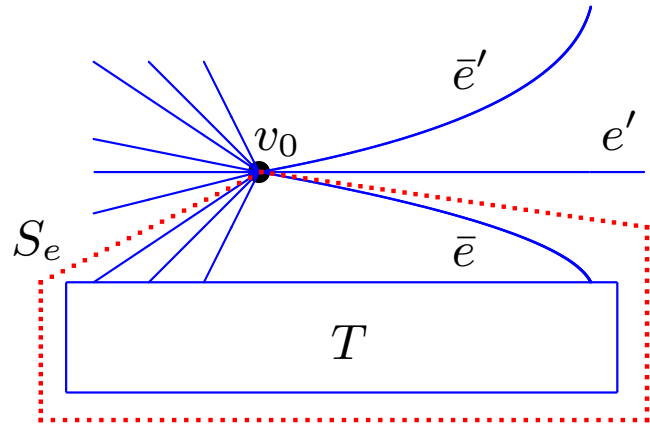
**Proposition 7** *Let  $D$  be a prime cut-point free knot-spoke diagram and let  $e$  be an edge incident to  $v_0$  and to another 4-valent vertex  $v_1$  such that  $D_e$  has a cut-point. Then there exists a simple closed curve  $S_e$  satisfying the following conditions.*

1.  $D_e \cap S_e = v_0$
2.  $S_e$  separates  $\bar{e}$  and  $\bar{e}'$  where the four edges incident to  $v_1$  in  $D$  are labeled with  $e, \bar{e}, e', \bar{e}'$  so that  $v_1$  is the crossing of the arcs  $e \cup e'$  and  $\bar{e} \cup \bar{e}'$ .
3.  $S_e$  separates  $D_e$  into two knot-spoke diagrams  $\bar{D}$  and  $\bar{D}'$  containing  $\bar{e}$  and  $\bar{e}'$ , respectively. Furthermore  $\bar{D}'$  is prime and cut-point free, and there is a sequence of non-spoke edges  $e_1, \dots, e_k$  of  $D$  not contained in  $\bar{D}'$  such that the knot-spoke diagram  $D_{e_1 e_2 \dots e_k}$  is identical with  $\bar{D}'$  on non-spoke edges in one side of  $S_e$  and has only spokes in the other side.

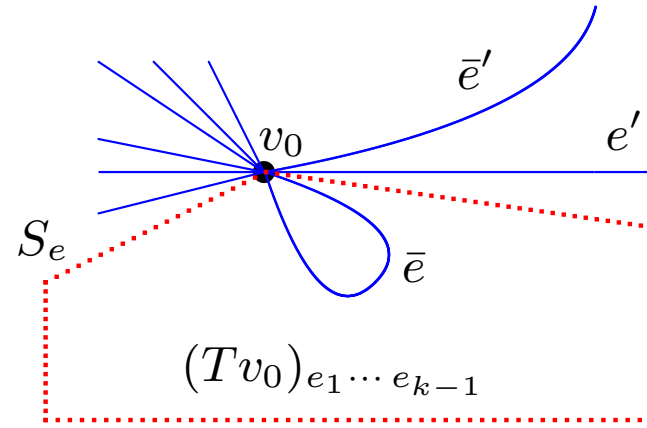
# Three cases for the proof of Proposition 7



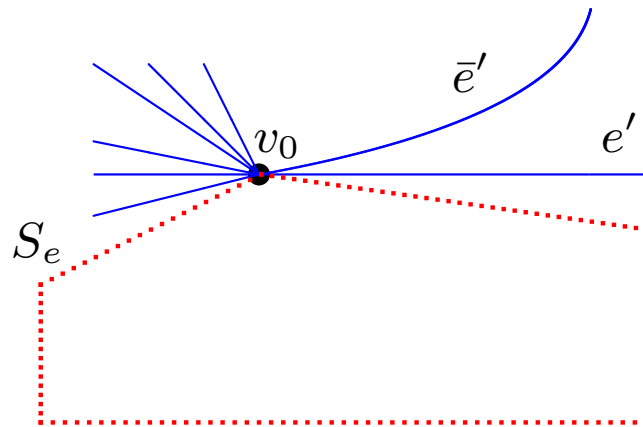
# Case 1.



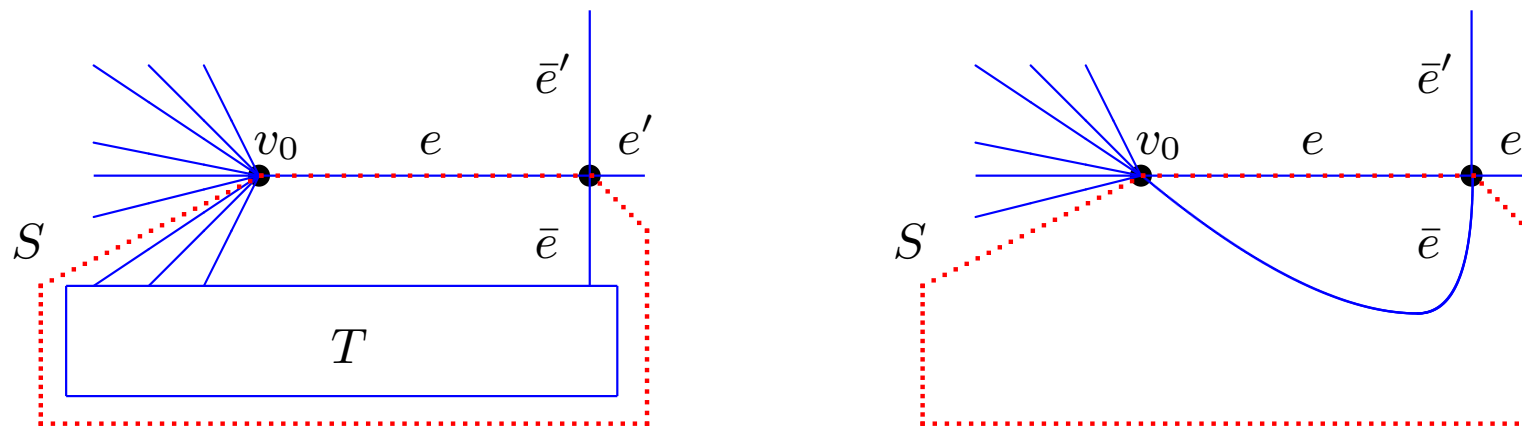
$D_e$



$D_{e e_1 \dots e_{k-1}}$

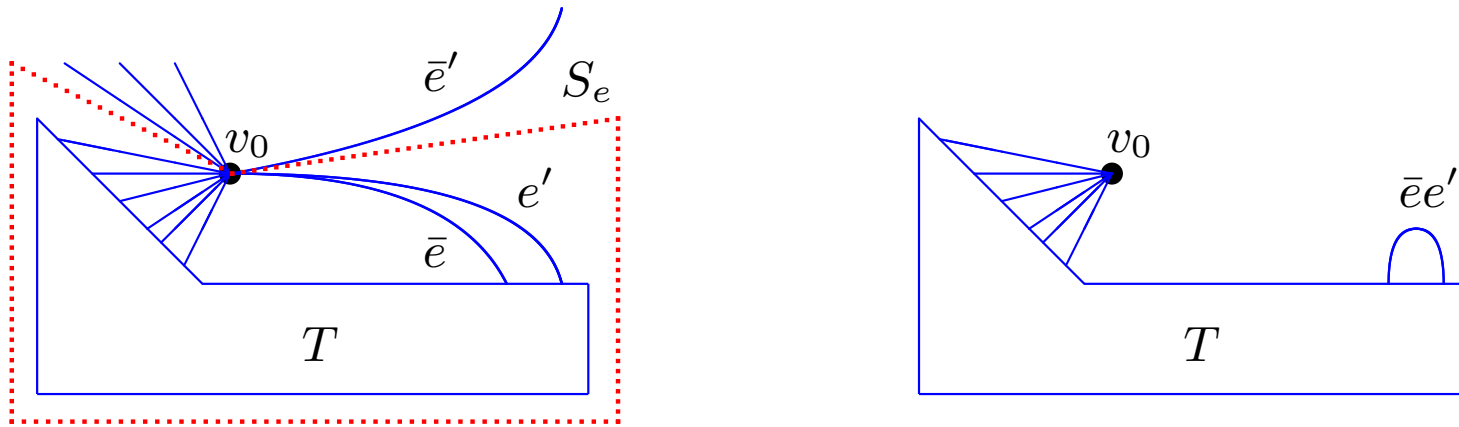


$D_{e_1 \dots e_{k-1} e}$

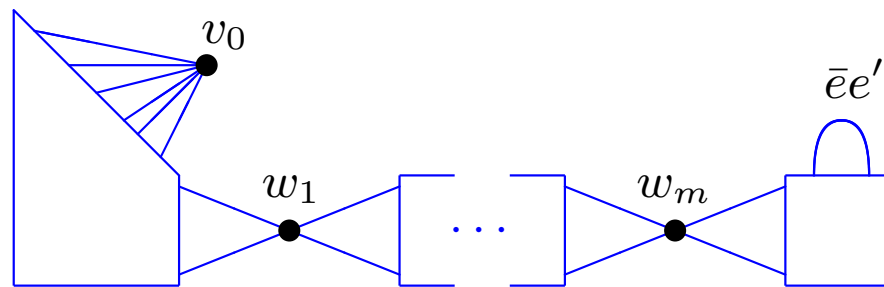


$D$  and  $D_{e_1 \dots e_{k-1}}$

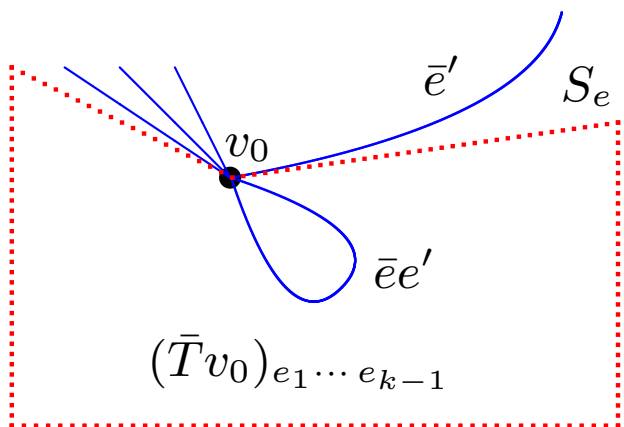
## Case 2.



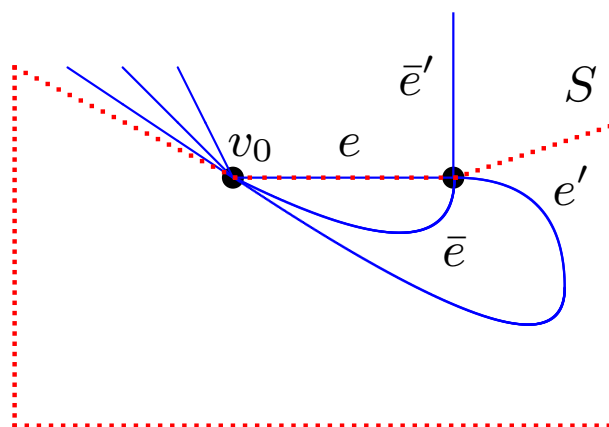
Case 2 :  $D_e$  and  $\bar{T}v_0$



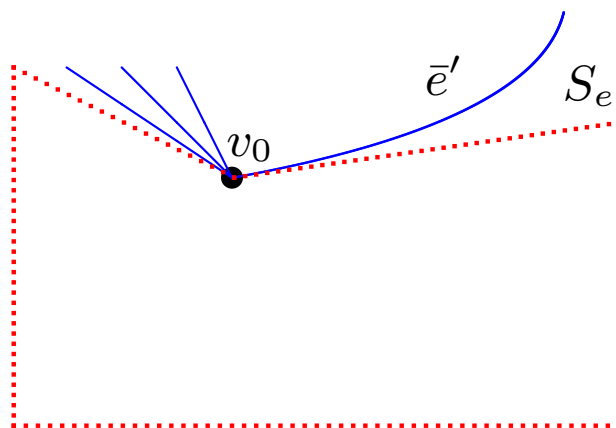
$\bar{T}v_0$  may have cut-points  $w_1, \dots, w_m$ .



$(\bar{T}v_0)_{e_1 \cdots e_{k-1}}$

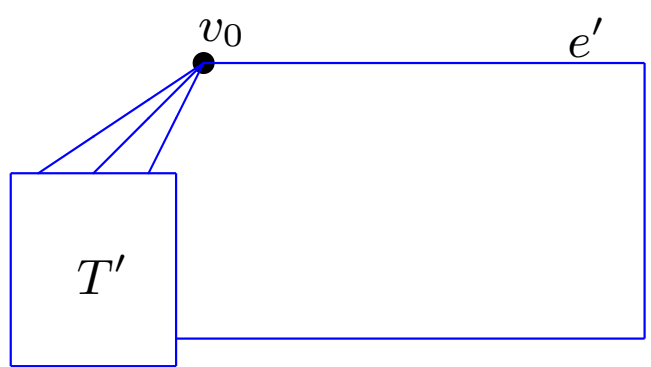
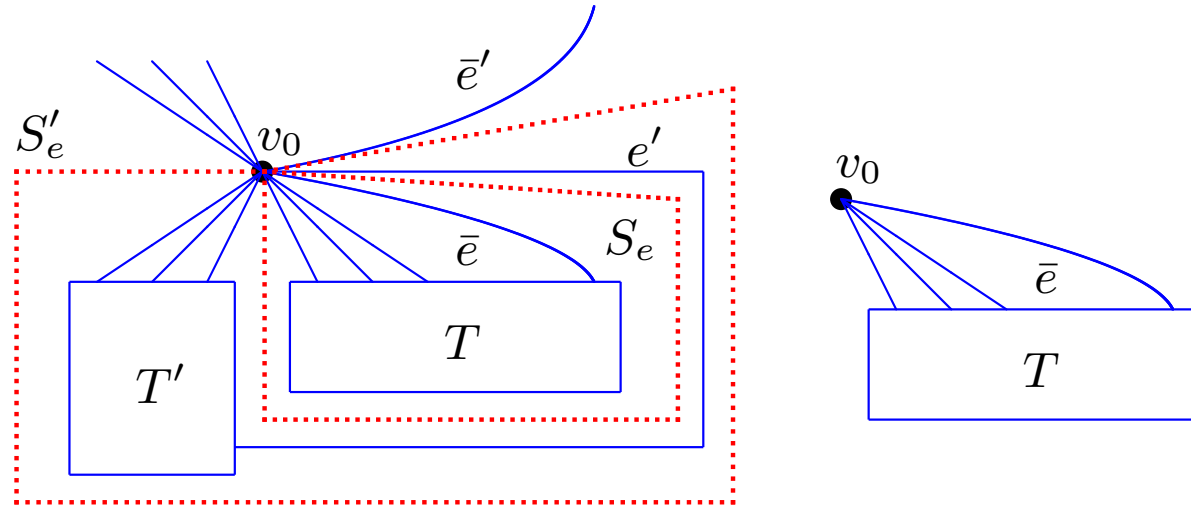


$(\bar{T}v_0)_{e_1 \cdots e_{k-1}}$  and  $D_{e_1 \cdots e_{k-1}}$



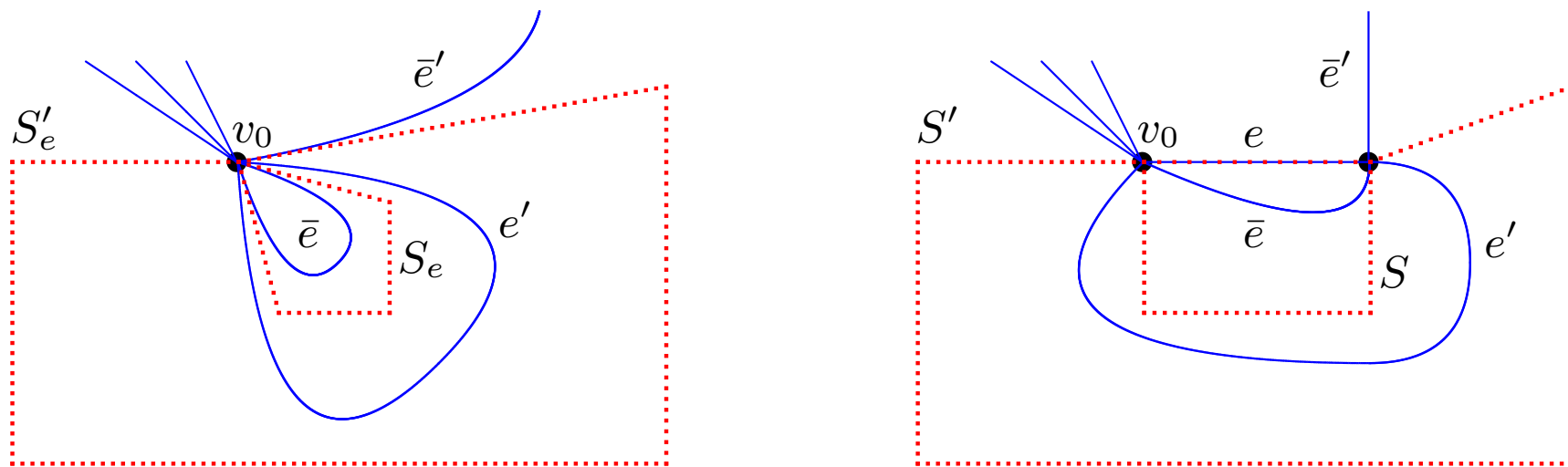
$D_{e_1 \cdots e_{k-1}} \bar{e}$

### Case 3.

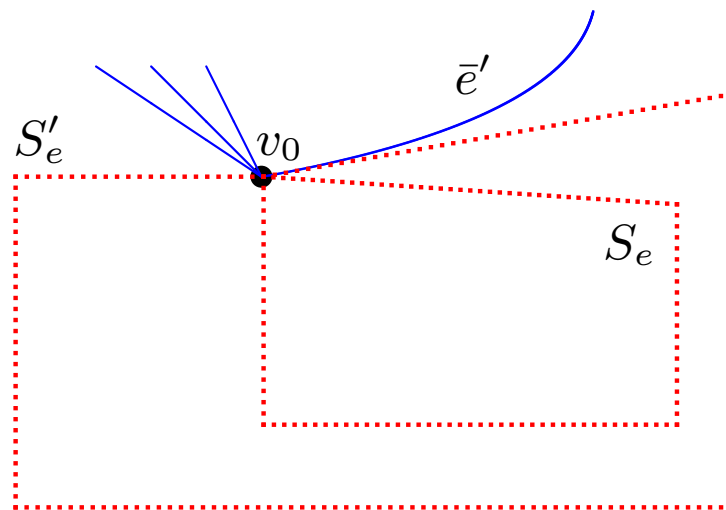


$D_e, Tv_0$  and  $T'v_0$





$(Tv_0 \cup T'v_0)_{e_1 \dots e_{k-1}}$  and  $D_{e_1 \dots e_{k-1}}$



$D_{e_1 \dots e_{k-1}} \bar{e}$

## Proof of Theorem 5

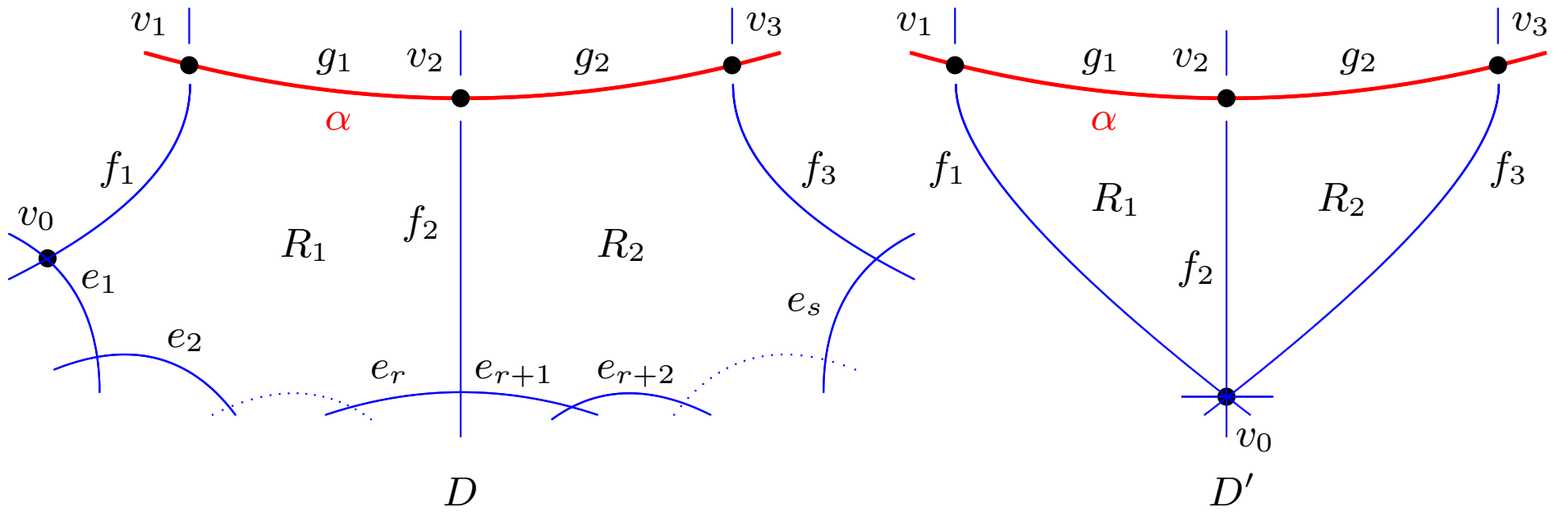
One of the three following cases must occur.

**Case I.** There is an arc of  $D$  which crosses over (or under) three times consecutively.

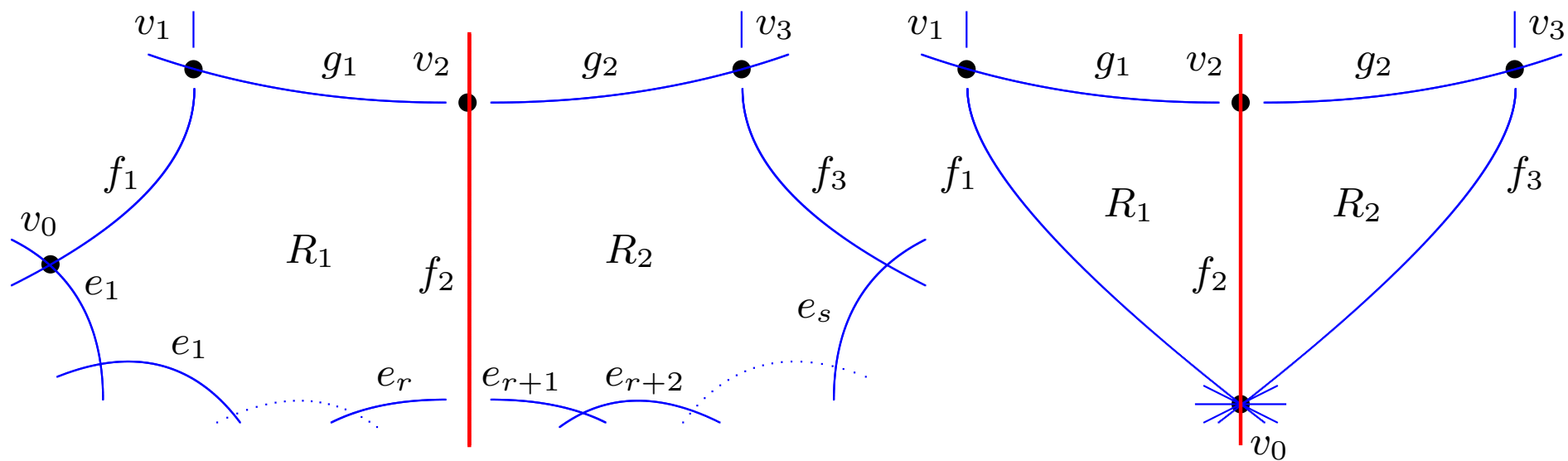
**Case II.** The two edges of an arc crossing one end of a non-alternating edge are both alternating.

**Case III.** One of the two edges of an arc crossing one end of a non-alternating edge is alternating and the other is non-alternating.

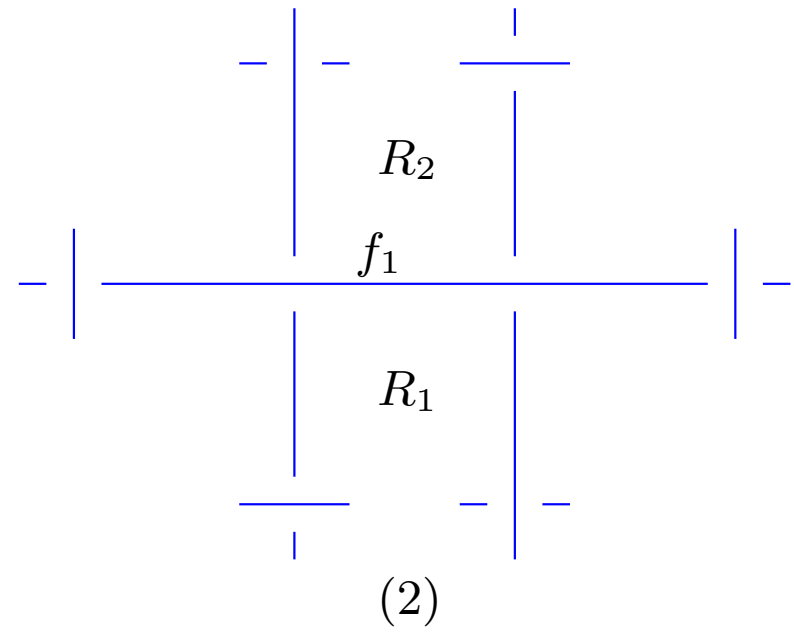
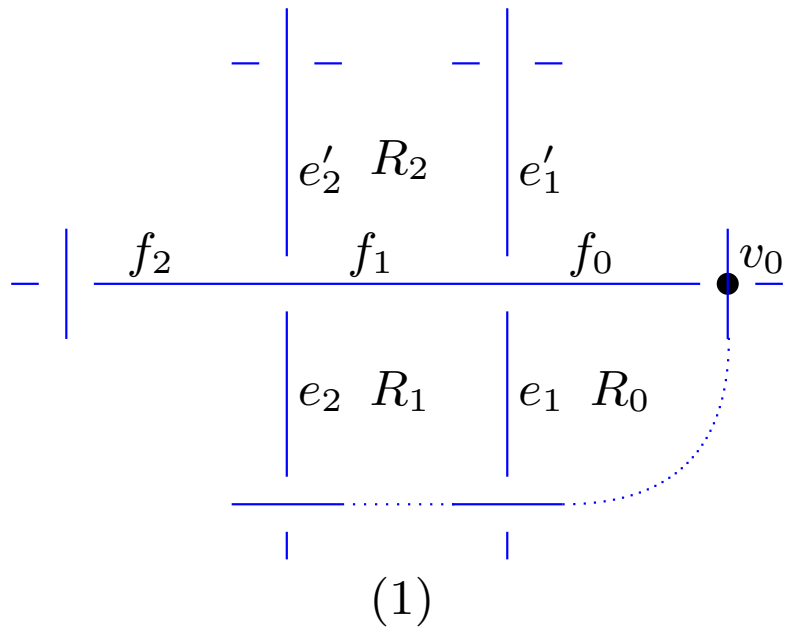
# Case I.

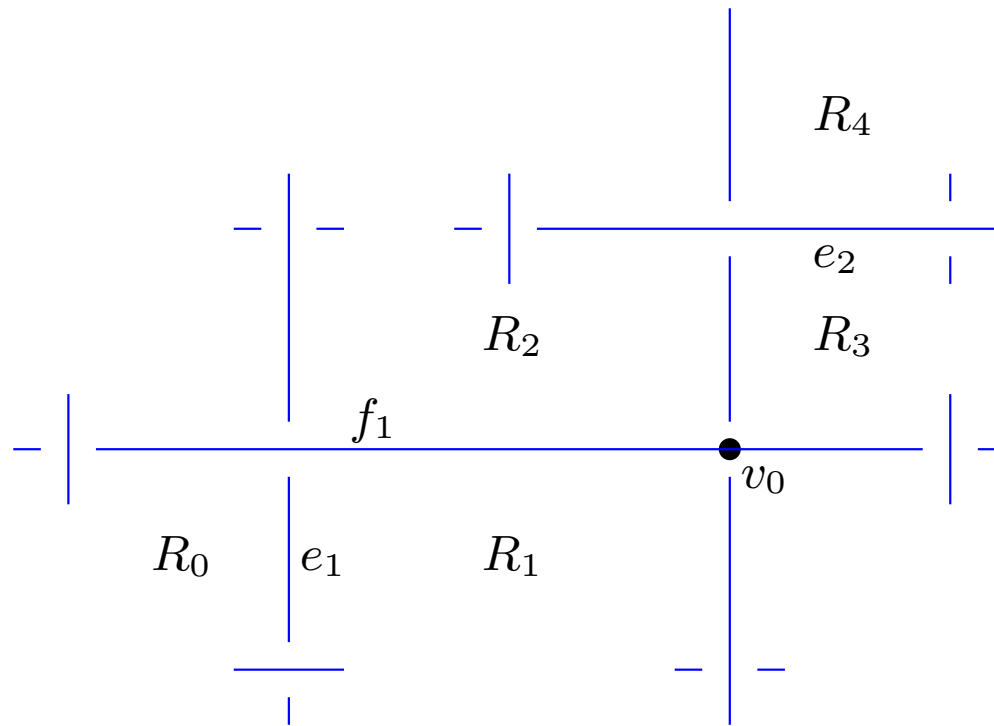


## Case II.



### Case III.





Case III (2)

**Thank you!**

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