On Maximal Collections of Essential Annuli in a Handlebody

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1 Backgroud

Let H_n be an orientable handlebody of genus n. A properly embedded surface S in H_n is essential if S is incompressible and no component of S is ∂-parallel in H_n.

It is well known that the only essential surface in H_1 consists of only parallel copies of a meridian disk of H_1, and any essential surface in H_n $(n \geq 2)$ either is ∂-compressible or totally contains essential disks. Thus an essential annulus in H_n with $n \geq 2$ may be regarded as a union of an essential disk and a band in H_n.

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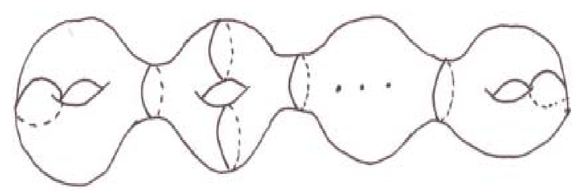
Let $\mathcal{D}$ be a collection of pairwise disjoint, non-parallel, essential disks in H_n. It is a fundamental fact that $\mathcal{D}$ contains only one disk if $n = 1$, and at most $3n - 3$ disks if $n \geq 2$.



Let $\mathcal{A}$ be a collection of pairwise disjoint, non-parallel, essential annuli in handlebody H_n. We say that $\mathcal{A}$ is maximal if A is an essential annulus in H_n with $A \cap \mathcal{A} = \emptyset$ then A is parallel to a component of $\mathcal{A}$ in H_n.

Question: How many annuli are there in a maximal collection of essential annuli in H_n?

It is a result of Rubinstein-Scharlemann that a maximal collection of essential annuli in H_2 may contain exactly 1, or 2, or at most 3 annuli.

Before stating Rubinstein-Scharlemann's result, we first review some definitions.

Definition

Suppose H is a handlebody and $C \subset \partial H$ is a simple closed curve. We say C is twisted if there is a properly embedded disk Δ in H such that a component of H cut along Δ is a solid torus T with $C \subset \partial T - \Delta$, and C is a (p, q)-torus knot on ∂T $(p \geq 2)$. We say C is a longitude if there exists an essential disk Δ in H such that C transversely meet Δ in a single point.

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Definition

Let H be a handlebody. Let A be a properly embedded annulus in H. If both boundary components of A are longitudes, A is called longitudinal; if both are twisted, A is called twisted.

Example

Twisted annulus: Let H_n be a handlebody of genus $n \geq 2$, Δ an essential disk in H_n which cuts out of a solid torus T from H_n. Let C be a (p, q)-torus knot on ∂T $(p \geq 2)$. Let A be a ∂-parallel annulus in T such that each component of ∂A is parallel to C on ∂T and A is parallel to the annulus in ∂T bounded by ∂A which contains the cutting section of Δ. Then A is twisted in H_n.

Longitudinal annulus:

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Rubinstein-Scharlemann Theorem:

Let $\mathcal{A}$ be a maximal collection of essential annuli in H_2. Then there exists an essential disk Δ in H_2 with $\Delta \cap \mathcal{A} = \emptyset$. Moreover,

(1) if Δ is separating in H_2, say, into two solid tori T_1 and T_2, then $\mathcal{A}$ contains two annuli A_1, A_2, such that A_1 is twisted lying in T_1, A_2 is twisted lying in T_2, and in each T_i, Δ is lying in the interior of the annulus in ∂T_i to which A_i is parallel in T_i, see figure 1.

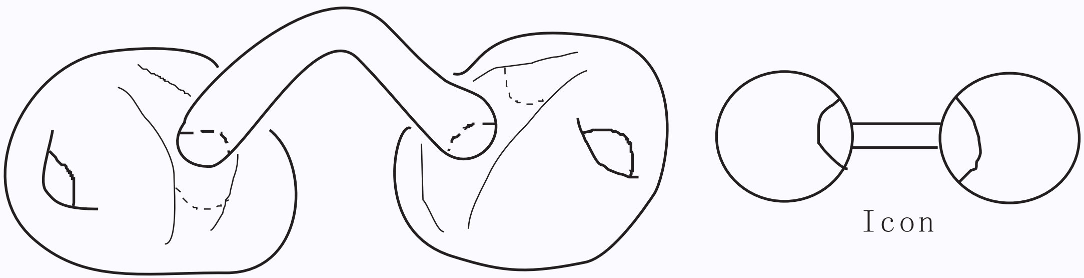

Figure 1

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(2) if Δ is non-separating in H_2, let T be the solid torus obtained by cutting H_2 open along Δ, then there are two subcases:

(2.1) $\mathcal{A}$ contains exact one longitudinal (therefore non-separating in H_2) annulus A in T such that the two cutting sections of Δ are lying in the interior of the two annuli in ∂T bounded by ∂A, see figure 2;

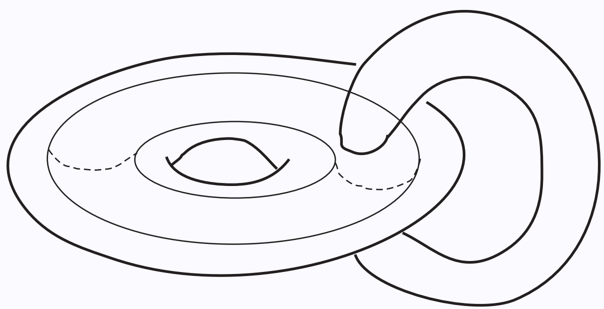
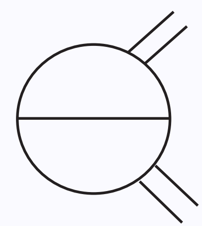


Figure 2



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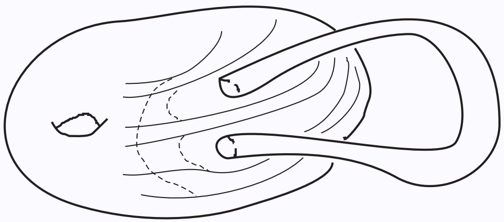
(2.2) $\mathcal{A}$ consists of exactly three twisted annuli A_0, A_1, A_2 in T, such that A_1, A_2 are lying in the solid torus T' via which A_0 is parallel to the annulus A' bounded by ∂A_0 in ∂T, and $A_1, A_2 \subset T'$ are parallel to two disjoint annuli on A' bounded by $\partial A_1, \partial A_2$ respectively, each of which contains one cutting section of Δ, see figure 3.



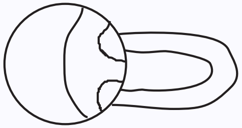


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Figure 3

We will generalize Rubinstein-Scharlemann's theorem to the case of H_n with $n \geq 3$.

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2 Main results

We use $|\cdot|$ to denote the number of elements of the corresponding set.

Theorem 1

Let $\mathcal{A}$ be a maximal collection of essential annuli in H_n with $n \geq 3$. Then $2 \leq |\mathcal{A}| \leq 4n-5$, and the bounds are best possible.

Theorem 2

Let H_n be a handlebody of genus $n \geq 3$. Then for each m, $2 < m < 4n-5$, there exists a maximal collection of essential annuli in H_n which contains exactly m annuli.

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3 Annulus-busting curves on ∂H_n

A simple closed curve C on ∂H_n which intersects every essential annulus in H_n nonempty is called an annulus-busting curve, or simply, an AB curve.

The next theorem shows the existence of the annulus-busting curves on ∂H_n ($n \geq 2$), which might be of interesting itself:

Theorem 3: For each $n \geq 2$, there exists infinitely many AB curves C on ∂H_n.

We will use the theorem to show the lower bound in Theorem 1.

In proving the theorem, we use a theorem of Hempel on Heegaard distance.

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Hempel's idea of the distance of a Heegaard splitting:

Let F be an orientable connected closed surface, α, β are two essential simple closed curves on F. Then there exists a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta$ on F such that, for each i, $1 \leq i \leq n$, α_{i-1} and α_i are pairwise disjoint. n is called the length of the sequence. The *distance* $d(\alpha, \beta)$ of α and β is defined to be the smallest length $n \in N$ of all sequences as above.

Let $V_1 \cup_F V_2$ be a Heegaard splitting. Denote by $D(V_1 \cup_F V_2)$ or $D(F)$ the integer $min\{d(C_1, C_2)|C_i$ bounds an essential disk in $C_i, i = 1, 2\}$, and call it the *distance* of the splitting $V_1 \cup_F V_2$.

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Hempel's Theorem

For any positive integers $m, n \geq 2$, there exists a Heegaard splittings $V_1 \cup_F V_2$ of genus n for a closed orientable 3-manifolds M with distance $D(F) > m$.

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Proof of the AB curve existence theorem:

By Hempel's theorem, there exists a Heegaard splittings $V_1 \cup_{F'} V_2$ of genus $n \geq 2$ for a closed orientable 3-manifolds M' with distance $D(F') \geq 3$, for any positive integers $n \geq 2$. Let C be a meridian curve of V_2. Let M be the 3-manifold obtained by adding a 2-handle to V_1 along C. Push F' slightly into the interior of M by isotopy, we get a surface F which is in fact a Heegaard surface in M. Clearly, $D(F) \geq D(F') \geq 3$.

Let A be an essential annulus properly embedded in H_n with $A \cap C = \emptyset$. We can show that A is essential in M.

On the other hand, since $D(F) \geq 3$, we can show that M contains no essential annulus (and torus), a contradiction.

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4 Proof of Theorem 1

In Rubinstein-Scharlemann's theorem, the icons are used to show the maximal collections of essential annuli in H_2 in a very simple and clear way. We will use similar icons in general cases.

Case $|\mathcal{A}| \geq 2$:

Let H^1, H^2 be two handlebodies of genus n_1, n_2, respectively, and $n_1 \geq 1$, and $n_2 \geq 2$, $n_1 + n_2 = n \geq 3$. Choose a simple closed curve C_1 on ∂H^1 in the following way: when $n_1 = 1$, let C_1 be a twisted curve on ∂H^1; when $n_1 > 1$, let C_1 be an annulus-busting curve on ∂H^1. Let C_2 be an annulus-busting curve on ∂H^2. Let A_i be a ∂-parallel properly embedded annulus in H^i such that each component of ∂A_i is parallel to C_i on ∂H^i, $i = 1, 2$.

Let D_i be a disk in the interior of the annulus bounded by ∂A_i on ∂H^i. Glue H^1 and H^2 together by identifying D_1 and D_2 to obtain a handlebody $H_n = H^1 \bigcup_{D_1=D_2} H^2$ of genus n. Then there is no other essential annulus in H_n which is disjoint from $A_1 \cup A_2$. Thus $\{A_1, A_2\}$ is maximal.

Case $|\mathcal{A}| \leq 4n-5$:

The proof here goes by induction on genus n of H_n.

Next we construct a maximal collection of essential annuli in H_n ($n \geq 3$) with exact $4n-5$ annuli.

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Let D be a disk, $\{a_1, a_2, \cdots, a_{4n-5}\}$ ($n \geq 2$) a collection pairwise disjoint simple arcs properly embedded in D shown as in figure 4. Let $T = D \times S^1$, and $A_i = a_i \times S^1$, $1 \leq i \leq 4n-5$. For each i, $1 \leq i \leq 2n-1$, A_i is parallel to an annulus A_i' bounded by ∂A_i in ∂T whose interior contains no component of $\partial\{A_i : 1 \leq i \leq 4n-5\}$. Let H be the handlebody of genus n obtained by adding $n-1$ 1-handle to T such that each A_i' $(1 \leq i \leq 2n-2)$ contains exact one end disk of the $n-1$ 1-handles.

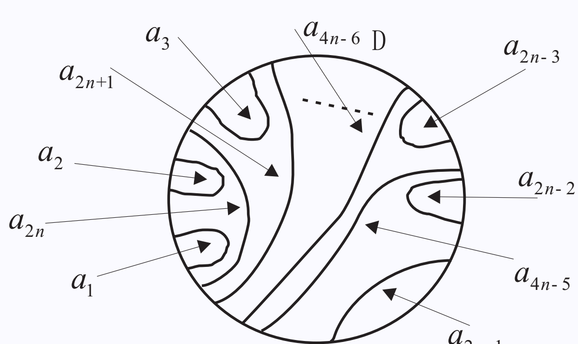
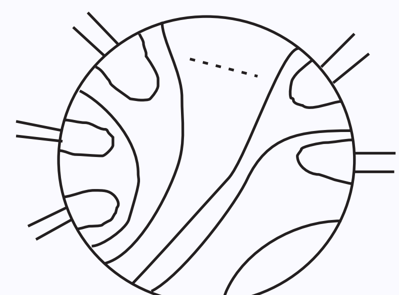


Figure 4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Let T' be another solid torus, $A' \subset \partial T'$ be an annulus, each of whose boundary components is a (p, q)-torus knot on $\partial T'$, $p \geq 2$. Union T and T' via a homeomorphism from A'_{2n-1} to A', we again get a handlebody H_n of genus n.

We can check that $\mathcal{A} = \{A_1, A_2, \cdots, A_{4n-5}\}$ is a maximal collection of pairwise disjoint non-parallel essential annuli in H_n.

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5 Proof of Theorem 2

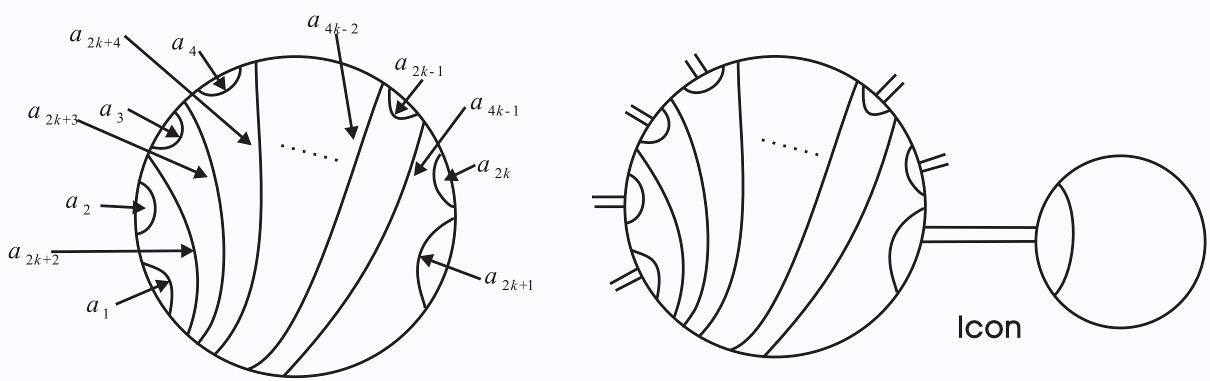
We only need to consider the case $2 < m < 4n - 5$. We will divide it into 6 cases to discuss, and in each case we will describe a maximal collection of essential annuli in H_n which contains exact m annuli.

- **Case 1**. $m = 4k, (1 \leq k \leq n - 2, k \in Z)$
- **Case 2**. $m = 4k + 2, (1 \leq k \leq n - 2, k \in Z)$
- **Case 3**. $m = 3$
- **Case 4**. $m = 5$
- **Case 5**. $m = 4k - 1, (2 \leq k \leq n - 2)$
- **Case 6**. $m = 4k + 1, (2 \leq k \leq n - 2)$

We only show the proof of case 1 here.

Proof of Case 1. $m = 4k$, $(1 \leq k \leq n-2, k \in Z)$.

Let D_0 be a disk. For $k \in \{1, 2, \cdots, n-2\}$, let $\alpha_1, \alpha_2, \cdots, \alpha_{4k-1}$ be a collection pairwise disjoint simple arcs properly embedded in D_0 as shown in Figure below.



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Let $T = D_0 \times S^1$, $A_i = \alpha_i \times S^1, 1 \leq i \leq 4k-1$. For $1 \leq i \leq 2k+1$, let A_i' be the annulus bounded by ∂A_i in ∂T with $A_i' \cap \partial A_j = \emptyset$ for any $j \neq i$, $1 \leq j \leq 4k-1$. Let H be the handlebody of genus $k+1$ obtained by adding k 1-handles to T such that each A_i' $(1 \leq i \leq 2k)$ contains exactly one end disk of the k 1-handles.

Let H' be a genus $n-k-1$ handlebody. Choose a simple closed curve C on $\partial H'$ in the following way: when $n-k-1 > 1$, C is annulus-busting in H'; when $n-k-1 = 1$, C is twisted in H'. Let A' be an annulus in $\partial H'$ such that each component of $\partial A'$ is parallel to C on $\partial H'$, and A'' a properly embedded annulus in H' such that $\partial A'' = \partial A'$ and A' and A'' are parallel in H'.

Let D' be a disk in the interior of A', and D a disk in the interior of A'_{2k+1} on ∂H. Glue H and H' together by identifying D and D' to obtain $H_n = H \bigcup_{D=D'} H'$.

We can check that $\mathcal{A} = \{A_1, A_2, \cdots, A_{4k-1}, A''\}$ is a maximal collection of essential annuli in H_n.

The proofs of other cases are similar.

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6 Two Questions

Question 1: Classify the maximal collections of essential annuli in H_n for $n \geq 3$.

Question 2: Let $\mathcal{A}$ be a maximal collection of pairwise disjoint, non-parallel, essential, m-punctured 2-spheres. Estimate $|\mathcal{A}|$.

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THANKS！