On Maximal Collections of Essential Annuli in a Handlebody

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The 4th East Asian Knot School, Tokyo University, Jan 20-25, 2008
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1 Backgroud

Let $H_n$ be an orientable handlebody of genus $n$. A properly embedded surface $S$ in $H_n$ is essential if $S$ is incompressible and no component of $S$ is $\partial$-parallel in $H_n$.

It is well known that the only essential surface in $H_1$ consists of only parallel copies of a meridian disk of $H_1$, and any essential surface in $H_n$ ($n \geq 2$) either is $\partial$-compressible or totally contains essential disks. Thus an essential annulus in $H_n$ with $n \geq 2$ may be regarded as a union of an essential disk and a band in $H_n$. 
Let $D$ be a collection of pairwise disjoint, non-parallel, essential disks in $H_n$. It is a fundamental fact that $D$ contains only one disk if $n = 1$, and at most $3n - 3$ disks if $n \geq 2$.

Let $A$ be a collection of pairwise disjoint, non-parallel, essential annuli in handlebody $H_n$. We say that $A$ is maximal if $A$ is an essential annulus in $H_n$ with $A \cap A = \emptyset$ then $A$ is parallel to a component of $A$ in $H_n$.

**Question: How many annuli are there in a maximal collection of essential annuli in $H_n$?**
It is a result of Rubinstein-Scharlemann that a maximal collection of essential annuli in $H_2$ may contain exactly 1, or 2, or at most 3 annuli.

Before stating Rubinstein-Scharlemann’s result, we first review some definitions.

**Definition**

Suppose $H$ is a handlebody and $C \subset \partial H$ is a simple closed curve. We say $C$ is **twisted** if there is a properly embedded disk $\Delta$ in $H$ such that a component of $H$ cut along $\Delta$ is a solid torus $T$ with $C \subset \partial T - \Delta$, and $C$ is a ($p, q$)-torus knot on $\partial T$ ($p \geq 2$). We say $C$ is a **longitude** if there exists an essential disk $\Delta$ in $H$ such that $C$ transversely meet $\Delta$ in a single point.
Definition
Let $H$ be a handlebody. Let $A$ be a properly embedded annulus in $H$. If both boundary components of $A$ are longitudes, $A$ is called **longitudinal**; if both are twisted, $A$ is called **twisted**.

Example
Twisted annulus: Let $H_n$ be a handlebody of genus $n \geq 2$, $\Delta$ an essential disk in $H_n$ which cuts out of a solid torus $T$ from $H_n$. Let $C$ be a $(p, q)$-torus knot on $\partial T$ ($p \geq 2$). Let $A$ be a $\partial$-parallel annulus in $T$ such that each component of $\partial A$ is parallel to $C$ on $\partial T$ and $A$ is parallel to the annulus in $\partial T$ bounded by $\partial A$ which contains the cutting section of $\Delta$. Then $A$ is twisted in $H_n$.

Longitudinal annulus:
Rubinstein-Scharlemann Theorem:

Let $\mathcal{A}$ be a maximal collection of essential annuli in $H_2$. Then there exists an essential disk $\Delta$ in $H_2$ with $\Delta \cap \mathcal{A} = \emptyset$. Moreover,

(1) if $\Delta$ is separating in $H_2$, say, into two solid tori $T_1$ and $T_2$, then $\mathcal{A}$ contains two annuli $A_1, A_2$, such that $A_1$ is twisted lying in $T_1$, $A_2$ is twisted lying in $T_2$, and in each $T_i$, $\Delta$ is lying in the interior of the annulus in $\partial T_i$ to which $A_i$ is parallel in $T_i$, see figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}
(2) if $\Delta$ is non-separating in $H_2$, let $T$ be the solid torus obtained by cutting $H_2$ open along $\Delta$, then there are two subcases:

(2.1) $\mathcal{A}$ contains exact one longitudinal (therefore non-separating in $H_2$) annulus $A$ in $T$ such that the two cutting sections of $\Delta$ are lying in the interior of the two annuli in $\partial T$ bounded by $\partial A$, see figure 2;
(2.2) \( \mathcal{A} \) consists of exactly three twisted annuli \( A_0, A_1, A_2 \) in \( T \), such that \( A_1, A_2 \) are lying in the solid torus \( T' \) via which \( A_0 \) is parallel to the annulus \( A' \) bounded by \( \partial A_0 \) in \( \partial T \), and \( A_1, A_2 \subset T' \) are parallel to two disjoint annuli on \( A' \) bounded by \( \partial A_1, \partial A_2 \) respectively, each of which contains one cutting section of \( \Delta \), see figure 3.

We will generalize Rubinstein-Scharlemann’s theorem to the case of \( H_n \) with \( n \geq 3 \).
2 Main results

We use $|\cdot|$ to denote the number of elements of the corresponding set.

Theorem 1
Let $\mathcal{A}$ be a maximal collection of essential annuli in $H_n$ with $n \geq 3$. Then $2 \leq |\mathcal{A}| \leq 4n - 5$, and the bounds are best possible.

Theorem 2
Let $H_n$ be a handlebody of genus $n \geq 3$. Then for each $m$, $2 < m < 4n - 5$, there exists a maximal collection of essential annuli in $H_n$ which contains exactly $m$ annuli.
3 Annulus-busting curves on $\partial H_n$

A simple closed curve $C$ on $\partial H_n$ which intersects every essential annulus in $H_n$ nonempty is called an annulus-busting curve, or simply, an AB curve.

The next theorem shows the existence of the annulus-busting curves on $\partial H_n$ ($n \geq 2$), which might be of interesting itself:

**Theorem 3:** For each $n \geq 2$, there exists infinitely many AB curves $C$ on $\partial H_n$.

We will use the theorem to show the lower bound in Theorem 1.

In proving the theorem, we use a theorem of Hempel on Heegaard distance.
Hempel’s idea of the distance of a Heegaard splitting:

Let $F$ be an orientable connected closed surface, $\alpha$, $\beta$ are two essential simple closed curves on $F$. Then there exists a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta$ on $F$ such that, for each $i, 1 \leq i \leq n$, $\alpha_{i-1}$ and $\alpha_i$ are pairwise disjoint. $n$ is called the length of the sequence. The distance $d(\alpha, \beta)$ of $\alpha$ and $\beta$ is defined to be the smallest length $n \in \mathbb{N}$ of all sequences as above.

Let $V_1 \cup_F V_2$ be a Heegaard splitting. Denote by $D(V_1 \cup_F V_2)$ or $D(F)$ the integer $\min\{d(C_1, C_2) | C_i$ bounds an essential disk in $C_i, i = 1, 2\}$, and call it the distance of the splitting $V_1 \cup_F V_2$. 
Hempel’s Theorem

For any positive integers $m, n \geq 2$, there exists a Heegaard splittings $V_1 \cup_F V_2$ of genus $n$ for a closed orientable 3-manifolds $M$ with distance $D(F) > m$. 
Proof of the AB curve existence theorem:

By Hempel’s theorem, there exists a Heegaard splittings $V_1 \cup_{F'} V_2$ of genus $n \geq 2$ for a closed orientable 3-manifolds $M'$ with distance $D(F') \geq 3$, for any positive integers $n \geq 2$. Let $C$ be a meridian curve of $V_2$. Let $M$ be the 3-manifold obtained by adding a 2-handle to $V_1$ along $C$. Push $F'$ slightly into the interior of $M$ by isotopy, we get a surface $F$ which is in fact a Heegaard surface in $M$. Clearly, $D(F) \geq D(F') \geq 3$.

Let $A$ be an essential annulus properly embedded in $H_n$ with $A \cap C = \emptyset$. We can show that $A$ is essential in $M$.

On the other hand, since $D(F) \geq 3$, we can show that $M$ contains no essential annulus (and torus), a contradiction.
4 Proof of Theorem 1

In Rubinstein-Scharlemann’s theorem, the icons are used to show the maximal collections of essential annuli in $H_2$ in a very simple and clear way. We will use similar icons in general cases.

Case $|\mathcal{A}| \geq 2$:

Let $H^1$, $H^2$ be two handlebodies of genus $n_1$, $n_2$, respectively, and $n_1 \geq 1$, and $n_2 \geq 2$, $n_1 + n_2 = n \geq 3$. Choose a simple closed curve $C_1$ on $\partial H^1$ in the following way: when $n_1 = 1$, let $C_1$ be a twisted curve on $\partial H^1$; when $n_1 > 1$, let $C_1$ be an annulus-busting curve on $\partial H^1$. Let $C_2$ be an annulus-busting curve on $\partial H^2$. Let $A_i$ be a $\partial$-parallel properly embedded annulus in $H^i$ such that each component of $\partial A_i$ is parallel to $C_i$ on $\partial H^i$, $i = 1, 2$. 
Let $D_i$ be a disk in the interior of the annulus bounded by $\partial A_i$ on $\partial H^i$. Glue $H^1$ and $H^2$ together by identifying $D_1$ and $D_2$ to obtain a handlebody $H_n = H^1 \cup_{D_1=D_2} H^2$ of genus $n$. Then there is no other essential annulus in $H_n$ which is disjoint from $A_1 \cup A_2$. Thus $\{A_1, A_2\}$ is maximal.

Case $|\mathcal{A}| \leq 4n - 5$:

The proof here goes by induction on genus $n$ of $H_n$.

Next we construct a maximal collection of essential annuli in $H_n$ ($n \geq 3$) with exact $4n - 5$ annuli.
Let $D$ be a disk, $\{a_1, a_2, \cdots, a_{4n-5}\}$ ($n \geq 2$) a collection pairwise disjoint simple arcs properly embedded in $D$ shown as in figure 4. Let $T = D \times S^1$, and $A_i = a_i \times S^1$, $1 \leq i \leq 4n - 5$. For each $i$, $1 \leq i \leq 2n - 1$, $A_i$ is parallel to an annulus $A'_i$ bounded by $\partial A_i$ in $\partial T$ whose interior contains no component of $\partial \{A_i : 1 \leq i \leq 4n - 5\}$. Let $H$ be the handlebody of genus $n$ obtained by adding $n - 1$ 1-handle to $T$ such that each $A'_i$ ($1 \leq i \leq 2n - 2$) contains exact one end disk of the $n - 1$ 1-handles.
Let $T'$ be another solid torus, $A' \subset \partial T'$ be an annulus, each of whose boundary components is a $(p, q)$-torus knot on $\partial T'$, $p \geq 2$. Union $T$ and $T'$ via a homeomorphism from $A'_{2n-1}$ to $A'$, we again get a handlebody $H_n$ of genus $n$.

We can check that $\mathcal{A} = \{A_1, A_2, \cdots, A_{4n-5}\}$ is a maximal collection of pairwise disjoint non-parallel essential annuli in $H_n$. 
5 Proof of Theorem 2

We only need to consider the case $2 < m < 4n - 5$. We will divide it into 6 cases to discuss, and in each case we will describe a maximal collection of essential annuli in $H_n$ which contains exact $m$ annuli.

- **Case 1.** $m = 4k$, $(1 \leq k \leq n - 2, k \in \mathbb{Z})$
- **Case 2.** $m = 4k + 2$, $(1 \leq k \leq n - 2, k \in \mathbb{Z})$
- **Case 3.** $m = 3$
- **Case 4.** $m = 5$
- **Case 5.** $m = 4k - 1$, $(2 \leq k \leq n - 2)$
- **Case 6.** $m = 4k + 1$, $(2 \leq k \leq n - 2)$
We only show the proof of case 1 here.

**Proof of Case 1.** \( m = 4k, \ (1 \leq k \leq n - 2, \ k \in \mathbb{Z}) \).

Let \( D_0 \) be a disk. For \( k \in \{1, 2, \cdots, n - 2\} \), let \( \alpha_1, \alpha_2, \cdots, \alpha_{4k-1} \) be a collection pairwise disjoint simple arcs properly embedded in \( D_0 \) as shown in Figure below.
Let $T = D_0 \times S^1, A_i = \alpha_i \times S^1, 1 \leq i \leq 4k - 1$. For $1 \leq i \leq 2k + 1$, let $A'_i$ be the annulus bounded by $\partial A_i$ in $\partial T$ with $A'_i \cap \partial A_j = \emptyset$ for any $j \neq i, 1 \leq j \leq 4k - 1$. Let $H$ be the handlebody of genus $k + 1$ obtained by adding $k$ 1-handles to $T$ such that each $A'_i$ $(1 \leq i \leq 2k)$ contains exactly one end disk of the $k$ 1-handles.

Let $H'$ be a genus $n - k - 1$ handlebody. Choose a simple closed curve $C$ on $\partial H'$ in the following way: when $n - k - 1 > 1$, $C$ is annulus-busting in $H'$; when $n - k - 1 = 1$, $C$ is twisted in $H'$. Let $A'$ be an annulus in $\partial H'$ such that each component of $\partial A'$ is parallel to $C$ on $\partial H'$, and $A''$ a properly embedded annulus in $H'$ such that $\partial A'' = \partial A'$ and $A'$ and $A''$ are parallel in $H'$. 
Let $D'$ be a disk in the interior of $A'$, and $D$ a disk in the interior of $A'_{2k+1}$ on $\partial H$. Glue $H$ and $H'$ together by identifying $D$ and $D'$ to obtain $H_n = H \cup_{D=D'} H'$.

We can check that $\mathcal{A} = \{A_1, A_2, \cdots, A_{4k-1}, A''\}$ is a maximal collection of essential annuli in $H_n$.

The proofs of other cases are similar.
6 Two Questions

Question 1: Classify the maximal collections of essential annuli in $H_n$ for $n \geq 3$.

Question 2: Let $\mathcal{A}$ be a maximal collection of pairwise disjoint, non-parallel, essential, $m$-punctured 2-spheres. Estimate $|\mathcal{A}|$. 
References


THANKS!