A unique decomposition theorem for tight contact 3-manifolds

Fan Ding and Hansjörg Geiges

1 Introduction

In this report, we present the main result of [2] and the rough idea of the proof.

All 3-manifolds are understood to be smooth and oriented. A closed connected 3-manifold is called *non-trivial* if it is not diffeomorphic to S^3. A non-trivial 3-manifold P is said to be *prime* if in every connected sum decomposition $P = P_0 \sharp P_1$ one of the summands P_0, P_1 is S^3. It is known that every non-trivial 3-manifold M admits a prime decomposition, i.e., M can be written as a connected sum of finitely many prime manifolds. Moreover, as shown by J. Milnor [4], the summands in this prime decomposition are unique up to order and diffeomorphism.

We prove the analogous result for tight contact 3-manifolds. A contact structure ξ on a 3-manifold M is a totally non-integrable 2-plane field. Our contact structures are understood to be cooriented and positive. This means that they can be defined as $\xi = \ker \alpha$ with a globally defined 1-form α satisfying the non-integrability condition that the 3-form $\alpha \wedge \mathrm{d}\alpha$ be a positive volume form.

A diffeomorphism $f : (M, \xi) \to (M', \xi')$ between contact manifolds is said to be a *contactomorphism* if its differential maps ξ to ξ' (preserving coorientations).

A contact structure ξ on a 3-manifold M is called *overtwisted* if there is an *embedded* 2-disc $\Delta \subset M$ tangent to ξ along the boundary, that is, with $T_p\Delta = \xi_p$ for all $p \in \partial\Delta$. A disc with this property is referred to as an *overtwisted disc*. A contact structure ξ is called *tight* if it is not overtwisted.

A fundamental result of Y. Eliashberg says that the classification of overtwisted contact structures reduces to a homotopical problem. The classification of tight contact structures, on the other hand, is a difficult problem having deep connections with 3-manifold topology. For instance, the standard contact structure

$$\xi_{st} := \ker(x\,\mathrm{d}y - y\,\mathrm{d}x + z\,\mathrm{d}t - t\,\mathrm{d}z)$$

on $S^3 \subset \mathbf{R}^4$ is the unique tight contact structure, up to isotopy, on S^3.

Given an embedded oriented surface S in a contact 3-manifold (M, ξ), the intersections $T_pS \cap \xi_p$, $p \in S$, define an oriented 1-dimensional foliation on S with singularities at the points p where the tangent plane T_pS coincides with ξ_p. This is called the *characteristic foliation* of S and is denoted by S_ξ. As shown by E. Giroux, the characteristic foliation S_ξ determines the germ of ξ near S. This allows one to glue contact manifolds along surfaces with diffeomorphic characteristic foliations.

Given an embedding $f : S \to (M, \xi)$, we write $S_{f^*\xi}$ for the induced characteristic foliation on S, that is, the pull-back to S via f of the characteristic foliation $(f(S))_\xi$.

Lemma 1 (V. Colin). *Let* (M, ξ) *be a tight contact* 3-*manifold. Given embeddings* $f_0, f_1 : S^2 \to M$, *there is a* *tight contact structure η on $S^2 \times [0,1]$ such that the characteristic foliation $(S^2 \times \{i\})_\eta$ coincides with $S^2_{f_i^*\xi}$, $i = 0, 1$. This contact structure η is unique up to isotopy rel boundary.*

Let $(M_0, \xi_0), (M_1, \xi_1)$ be two connected tight contact 3-manifolds. Equip the 3-disc D^3 with its standard orientation. Let $\phi_0 : D^3 \to M_0$, $\phi_1 : D^3 \to M_1$ be embeddings such that ϕ_0 reverses and ϕ_1 preserves orientation. Let η be the contact structure on $S^2 \times [0,1]$, constructed in the preceding lemma, with the property that $(S^2 \times \{i\})_\eta = (\partial D^3)_{\phi_i^*\xi_i}$, $i = 0, 1$. Then set

$$(M', \xi') = (M \backslash \mathrm{Int}(B), \xi) \cup_\partial (S^2 \times [0,1], \eta),$$

where $M = M_0 \sqcup M_1$, $B = \phi_0(D^3) \sqcup \phi_1(D^3)$, 'Int' stands for interior, and $\cup_\partial$ denotes the obvious gluing along the boundary. Topologically, M' is the connected sum $M_0 \sharp M_1$ of M_0 and M_1. We write $\xi_0 \sharp \xi_1$ for the contact structure ξ'. We also use the notation $(M_0, \xi_0)\sharp(M_1, \xi_1)$ for this connected sum of tight contact 3-manifolds. As shown by V. Colin [1], the contact structure $\xi_0 \sharp \xi_1$ on $M_0 \sharp M_1$ is tight and does not depend, up to contactomorphism, on the choice of embeddings ϕ_0, ϕ_1. This connected sum operation is commutative and associative. (S^3, ξ_{st}) serves as the zero element.

Theorem 2. *Every non-trivial tight contact* 3*-manifold* (M, ξ) *is contactomorphic to a connected sum*

$$(M_1, \xi_1)\sharp \cdots \sharp(M_k, \xi_k)$$

of finitely many prime tight contact 3*-manifolds. The summands* (M_i, ξ_i), $i = 1, \ldots, k$, *are unique up to order and contactomorphism.*

As shown by V. Colin [1], given a fixed connected sum decomposition $M = M_0 \sharp M_1$ of a 3-manifold M, for any tight contact structure ξ on M there are — up to isotopy — unique tight contact structures ξ_i on M_i, $i = 0, 1$, such that $\xi_0 \sharp \xi_1$ is the given contact structure ξ. The prime decomposition theorem for tight contact 3-manifolds is an immediate consequence.

Although Colin's result goes a long way, it is not quite strong enough to prove the *unique* decomposition theorem for tight contact 3-manifolds directly. This is due to the fact that the system of 2-spheres in a given manifold M defining the prime decomposition of M is not, in general, unique up to isotopy.

The argument for the unique decomposition of tight contact 3-manifolds given here closely follows the variant of Milnor's argument given in J. Hempel's book [3].

2 Proof of the unique decomposition theorem

There is a well-defined procedure for capping off a compact tight contact 3-manifold whose boundary consists of a collection of 2-spheres. Suppose that (M, ξ) is a tight contact 3-manifold whose boundary consists of a collection of 2-spheres. We use the notation $\widehat{M}$ for the manifold obtained from M by capping off each 2-sphere with a 3-cell. We can construct a tight contact structure $\widehat{\xi}$ on $\widehat{M}$, unique up to isotopy, such that $\widehat{\xi}|_M = \xi$.

A closed connected 3-manifold M is said to be *irreducible* if every embedded 2-sphere bounds a 3-disc in M. Irreducible 3-manifolds (except S^3) are prime. There is only one orientable prime 3-manifold that is not irreducible, namely, $S^2 \times S^1$.

Here is the rough idea of the proof of the unique decomposition theorem. Let

$$(M_1, \xi_1)\sharp \cdots \sharp(M_k, \xi_k)$$

and

$$(M_1^*, \xi_1^*)\sharp \cdots \sharp(M_l^*, \xi_l^*)$$

be two prime decompositions of a given non-trivial tight contact 3-manifold (M, ξ).

(i) Suppose some M_i (say M_k) is diffeomorphic to $S^2 \times S^1$. Then M contains a non-separating 2-sphere. One can find a non-separating 2-sphere in at least one of M_j^*'s, say M_l^*, which therefore must be a copy of $S^2 \times S^1$. By a theorem of Eliashberg, there is a unique tight contact structure on $S^2 \times S^1$. Thus, (M_k, ξ_k) is contactomorphic to (M_l^*, ξ_l^*). Also

$$(M_1, \xi_1)\sharp \cdots \sharp(M_{k-1}, \xi_{k-1})$$

and

$$(M_1^*, \xi_1^*)\sharp \cdots \sharp(M_{l-1}^*, \xi_{l-1}^*)$$

are contactomorphic.

(ii) Suppose all the M_i are irreducible. Then each M_j^* must be irreducible. Choose a separating 2-sphere $S \subset M$ such that the closures U, V of the components of $M \backslash S$ satisfy

$$(\widehat{U}, \widehat{\xi|_U}) = (M_1, \xi_1)\sharp \cdots \sharp(M_{k-1}, \xi_{k-1})$$

and $(\widehat{V}, \widehat{\xi|_V}) = (M_k, \xi_k)$.

Similarly, there exist pairwise disjoint 2-spheres $T_1, \ldots, T_{l-1}$ in M such that — with $W_1, \ldots, W_l$ denoting the closures of the components of $M \backslash (T_1 \cup \ldots \cup T_{l-1})$, and ξ_j the restriction of ξ to W_j — we have $(\widehat{W}_j, \widehat{\xi}_j) = (M_j^*, \xi_j^*)$, $j = 1, \ldots, l$.

Suppose that the system $T_1, \ldots, T_{l-1}$ of embedded spheres has been chosen in general position with respect to S and with

$S \cap (T_1 \cup \ldots \cup T_{l-1})$ having the minimal number of components among all such systems.

The minimality condition implies $S \cap (T_1 \cup \ldots \cup T_{l-1}) = \emptyset$. Thus, we have $S \subset W_j$ for some $j \in \{1, \ldots, l\}$. Since $\widehat{W_j} = M_j^*$ is irreducible, S bounds a 3-cell in M_j^*. Thus, S cuts W_j into two pieces X and Y, where say $\widehat{Y} = S^3$. We have $(\widehat{Y}, \widehat{\xi|_Y}) = (S^3, \xi_{st})$ and $(\widehat{X}, \widehat{\xi|_X}) = (M_j^*, \xi_j^*)$.

In the case $Y \subset V$, the numbering (including that of W_j) can be chosen in such a way that $W_1, \ldots, W_{j-1}, X \subset U$ and $Y, W_{j+1}, \ldots, W_l \subset V$, with $j \leq l-1$. (The case with $Y \subset U$ is analogous.) We conclude that

$$(M_1, \xi_1) \sharp \cdots \sharp (M_{k-1}, \xi_{k-1}) = (\widehat{U}, \widehat{\xi|_U})$$

$$= (\widehat{W}_1, \widehat{\xi}_1) \sharp \cdots \sharp (\widehat{W}_{j-1}, \widehat{\xi}_{j-1}) \sharp (\widehat{X}, \widehat{\xi|_X}) = (M_1^*, \xi_1^*) \sharp \cdots \sharp (M_j^*, \xi_j^*)$$

and

$$(M_k, \xi_k) = (\widehat{V}, \widehat{\xi|_V}) = (\widehat{Y}, \widehat{\xi|_Y}) \sharp (\widehat{W}_{j+1}, \widehat{\xi}_{j+1}) \sharp \cdots \sharp (\widehat{W}_l, \widehat{\xi}_l)$$
$$= (M_{j+1}^*, \xi_{j+1}^*) \sharp \cdots \sharp (M_l^*, \xi_l^*).$$

Since M_k is prime, we must have $j = l-1$, hence $(M_k, \xi_k) = (M_l^*, \xi_l^*)$.

In both cases (i) and (ii), the proof of the unique decomposition theorem concludes by induction on the number of prime summands.

References

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