An asymptotic behavior of the dilatation for a family of pseudo-Anosov braids

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pseudo-Anosov braid

D_n; n-punctured disk
$\mathcal{M}(D_n)$; mapping class group of D_n

One can classify mapping classes $\phi \in \mathcal{M}(D_n)$ into 3 types

1. the order of ϕ is finite
2. the order of ϕ is infinite and ϕ is reducible
3. (pseudo-Anosov) the order of ϕ is infinite and ϕ is irreducible

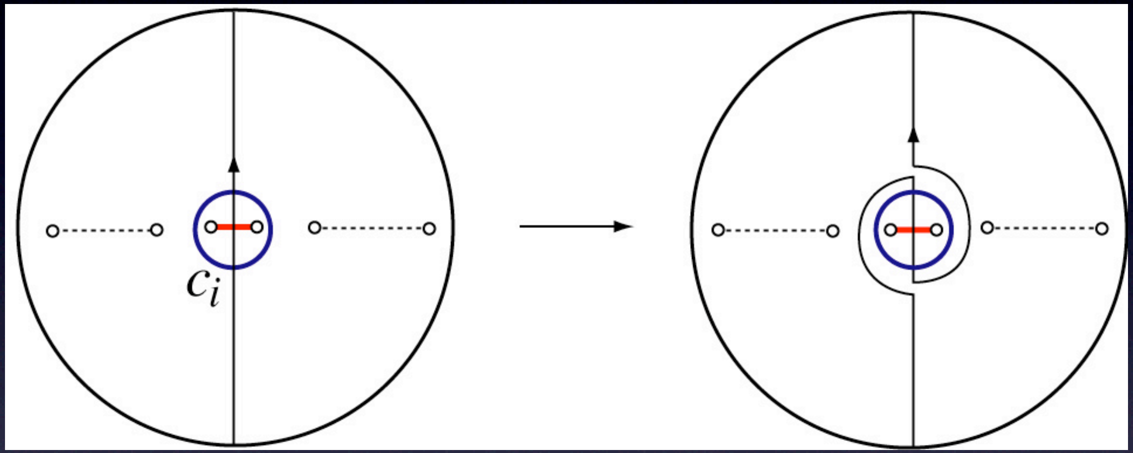
In case 3, ϕ contains a beautiful homeo. $\Phi : D_n \to D_n$ (so called pA homeo.) as a representative.

[To study $\mathcal{M}(D_n)$, it is very handy to represent mapping classes as geometric braids]

B_n; n-braid group

${}^{\exists}\Gamma : B_n \to \mathcal{M}(D_n)$; surjective homomorphism

$\sigma_i \mapsto t_i$ (t_i is the positive half twist)



Using Γ, one can classify braids into 3 types.

two invariants, dilatation and volume

pA braids have two invariants.

1. $\lambda(b) := \lambda(\Phi_b) > 1$; <u>dilatation</u>
2. $\mathrm{vol}(b) := \mathrm{vol}(\mathbb{T}(b)) > 0$; <u>volume</u>

Choose any representative $f : D_n \to D_n$ of $\Gamma(b) \in \mathcal{M}(D_n)$, and form the mapping torus

$$\mathbb{T}(b) := D_n \times [0,1]/\sim,$$

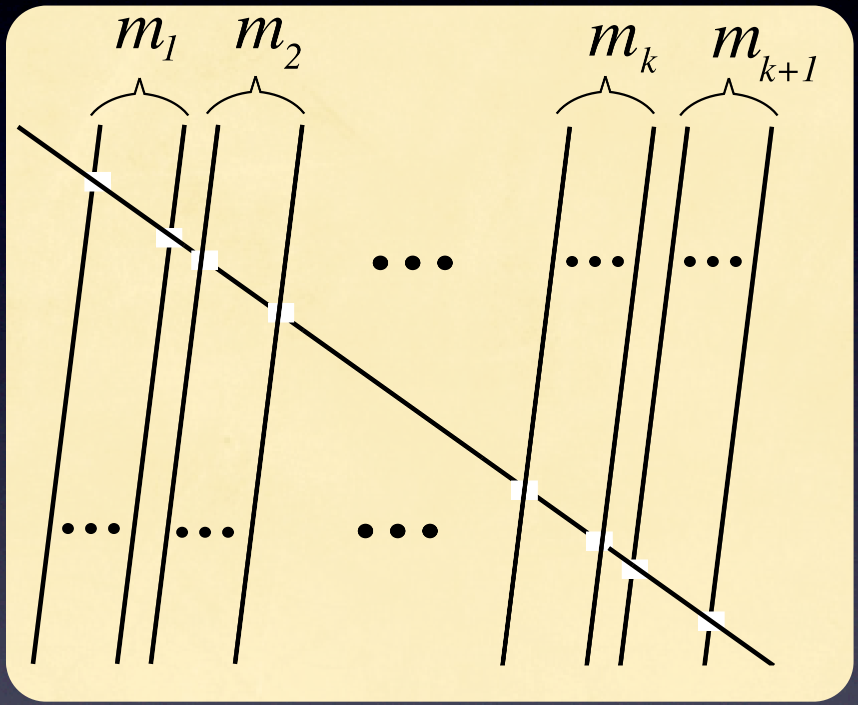
where $\sim$ identifies $(x,0)$ with $(f(x),1)$.

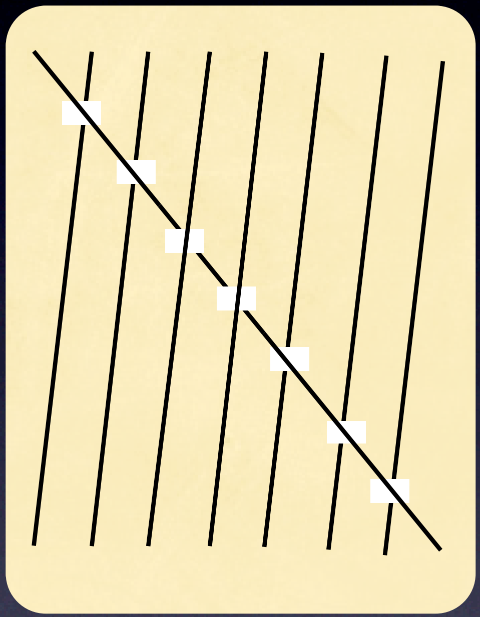
Thm. The braid b is pA $\iff$ The mapping torus $\mathbb{T}(b)$ admits a complete hyperbolic structure of finite volume.

If a braid is pA, we can think about its volume.

A family of braids

We consider the following braid $\beta_{(m_1,m_2,\cdots,m_{k+1})}$ for each integer $k \geq 1$ and each integer $m_i \geq 1$. These braids are all pA.





$\beta_{(2,2,3)}$

We would like to say this family is nice, because it has interesting properties on the two invariant.

1. Our braids have a nice inductive formula to compute their dilatation.

For an integral polynomial $f(t)$ of degree d, the *reciprocal* of $f(t)$, denoted by $f_*(t)$, is $t^d f(1/t)$.

Thm. A (Inductive formula) The dilatation of the pA braid $\beta_{(m_1,\cdots,m_{k+1})}$ is the largest root of the polynomial

$$t^{m_{k+1}} R_{(m_1,\cdots,m_k)}(t) + (-1)^{k+1} R_{(m_1,\cdots,m_k)_*}(t),$$

where $R_{(m_1,\cdots,m_i)}(t)$ is given inductively as follows:

$$\begin{aligned} R_{(m_1)}(t) &= t^{m_1+1}(t-1) - 2t, \text{ and} \\ R_{(m_1,\cdots,m_i)}(t) &= t^{m_i}(t-1) R_{(m_1,\cdots,m_{i-1})}(t) + (-1)^i 2t R_{(m_1,\cdots,m_{i-1})_*}(t) \text{ for } 2 \le i \le k. \end{aligned}$$

2. The dilatation of our braids can be arbitrarily small.

Thm B.(Asymptotic behavior)

$$\lim_{m_1,\cdots,m_{k+1}\to\infty} \lambda(\beta_{(m_1,\cdots,m_{k+1})}) = 1$$

The dilatation measures some complexity of pA mapping classes. Thm B. says that if all indices go to ∞, then the complexity of our braids goes to 0.

Question. What happen for the volume of a family of pA braids whose dilatation is arbitrarily small.

(Known two families $\{b_n\}$ and $\{\gamma_n\}$ with arbitrarily small dilatation)

1. $\lim_{n\to\infty} \mathrm{vol}(b_n) = \infty$ and the number of the cusps of the mapping torus $\mathbb{T}(b_n)$ goes to ∞ as n goes to ∞.

2. $\lim_{n\to\infty} \mathrm{vol}(\gamma_n) < \infty$ and the number of the cusps of the mapping torus $\mathbb{T}(\gamma_n)$ is bounded from above.

We can give another family of braids:

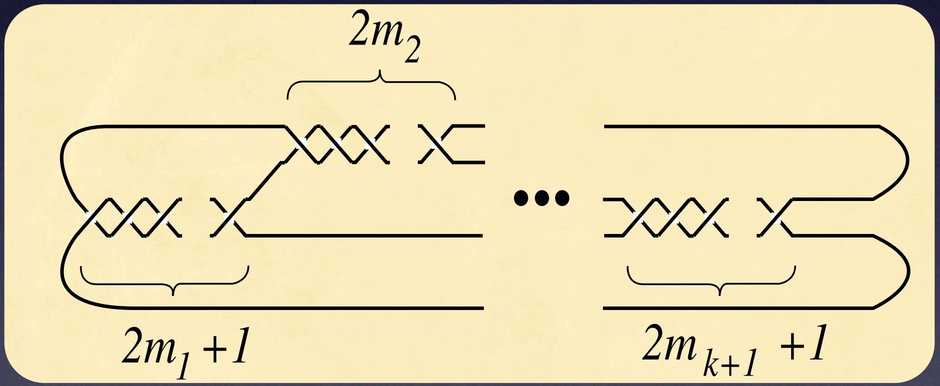
Thm. There exist a family of pA braids β_n such that

$$\lim_{n\to\infty} \lambda(\beta_n) = 1 \text{ and } \lim_{n\to\infty} \mathrm{vol}(\beta_n) = \infty$$

and such that the number of the cusps of the mapping torus $\mathbb{T}(\beta_n)$ is 2 for each n.

(Proof.) Note that the braided link $\overline{\beta_{(m_1,\cdots,m_{k+1})}}$ is an alternating link (in fact 2 bridge link) with twist number $k+1$. A result by Lackenby tells us that

$$\mathrm{vol}(\beta_{(m_1,\cdots,m_{k+1})}) > \frac{1}{2}(k-1)v_3$$



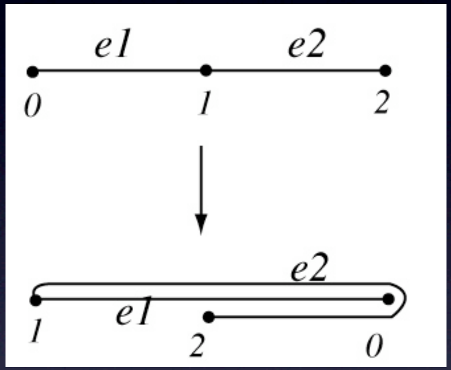
On the other hand, for any k, we can make the dilatation arbitrarily small:

$$\lim_{m_1,\cdots,m_{k+1}\to\infty} \lambda(\beta_{(m_1,\cdots,m_{k+1})}) = 1. \square$$

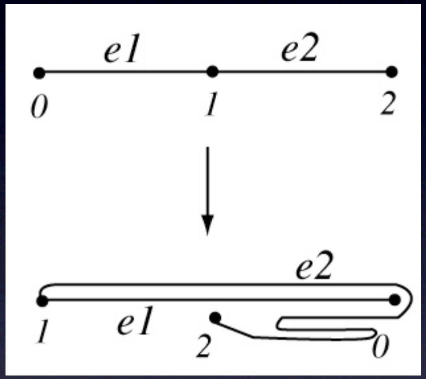
graph maps, transition matrices

$\mathcal{G}$; graph,
A continuous map $g : \mathcal{G} \to \mathcal{G}$ is called a graph map.

We suppose that g is "tight".



tight; OK



loose; NG

One can define the transition matrix $M(g) = (m_{i,j})$ as follows:

$m_{i,j}$:= the number of times that $g(e_j)$ passes through e_i.

Notation

For the transition matrix $M(g)$,

- $M(g)(t) := |tI - M(g)|$
  $= \text{characteristic poly. of the transition matrix } M(g)$

- $\lambda(M(g)) := \text{the spectral radius of the transition matrix } M(g)$
  $= \max\{|\lambda| \;;\; \lambda \text{ is an eigenvalue of } M(g)\}$

In general, the dilatation of a pA braid is given by the spectral radius of the transition matrix for some graph maps.

combined tree maps

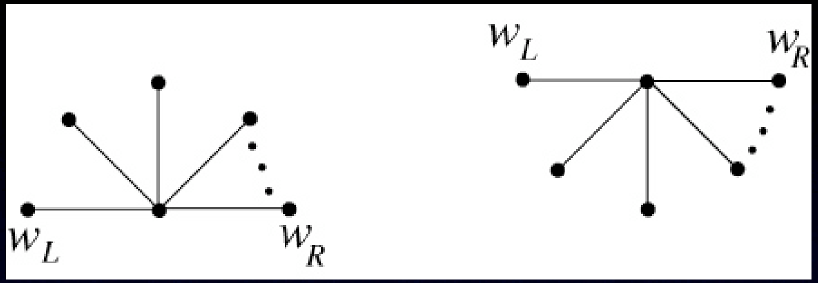
Inductively, we will define the combined tree $\mathcal{Q}_{(m_1,\cdots,m_{k+1})}$ and the combined tree map $q_{(m_1,\cdots,m_{k+1})} : \mathcal{Q}_{(m_1,\cdots,m_{k+1})} \to \mathcal{Q}_{(m_1,\cdots,m_{k+1})}$

Prop. (combined tree maps know the dilatation of $\beta_{(m_1,\cdots,m_{k+1})}$)

$$\lambda(\beta_{(m_1,\cdots,m_{k+1})}) = \lambda(M(q_{(m_1,\cdots,m_{k+1})})).$$

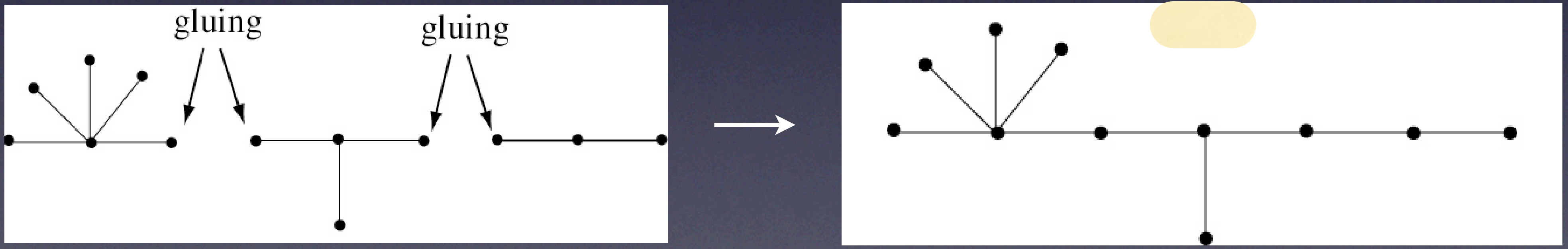
(combined trees)

Let $\mathcal{G}_{n,+}$ and $\mathcal{G}_{n,-}$ be trees of star type having one vertex of valence $n+1$.



$\mathcal{G}_{n,+}$ $\mathcal{G}_{n,-}$

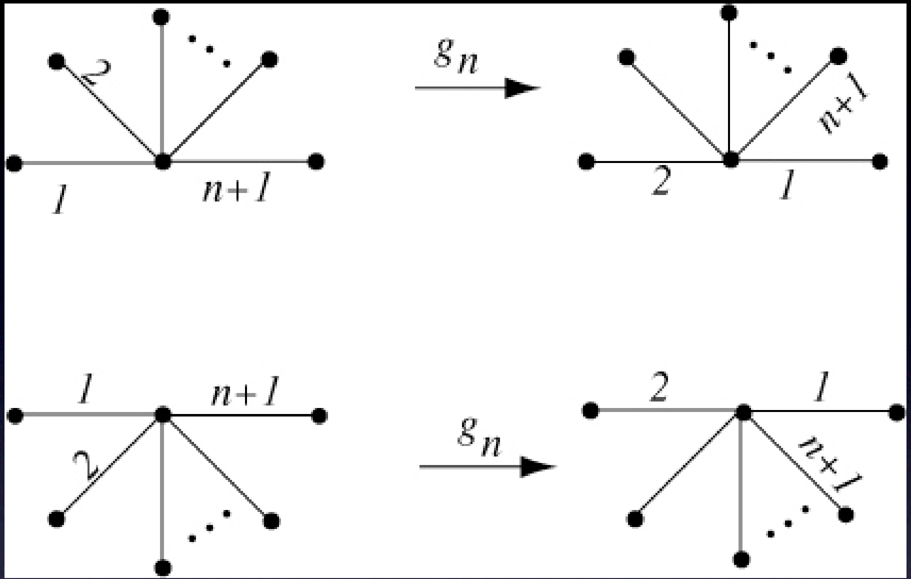
The combined tree $\mathcal{Q}_{(m_1,\cdots,m_{k+1})}$ is the one obtained by gluing $k+1$ trees $\mathcal{G}_{m_1,+}, \mathcal{G}_{m_2,-} \cdots, \mathcal{G}_{m_{k+1},(-1)^k}$ in the following way:



$(m_1, m_2, m_3) = (4, 2, 1)$ $\mathcal{Q}_{(4,2,1)}$

(combined tree maps)

First, take the following tree map $g_n : \mathcal{G}_{n,\pm} \to \mathcal{G}_{n,\pm}$:



Suppose that the combined tree map (up to k) is defined:

$$q_{(m_1,\cdots,m_k)} : \mathcal{Q}_{(m_1,\cdots,m_k)} \to \mathcal{Q}_{(m_1,\cdots,m_k)}.$$

Note that the two trees $\mathcal{Q}_{(m_1,\cdots,m_k)}$ and $\mathcal{G}_{m_{k+1}}$ can be thought as the subtrees of $\mathcal{Q}_{(m_1,\cdots,m_{k+1})}$.

Take the extension $\widehat{q}$ of $q_{(m_1,\cdots,m_k)} : \mathcal{Q}_{(m_1,\cdots,m_k)} \circlearrowleft$:

$$\widehat{q} : \mathcal{Q}_{(m_1,\cdots,m_{k+1})} \to \mathcal{Q}_{(m_1,\cdots,m_{k+1})}$$

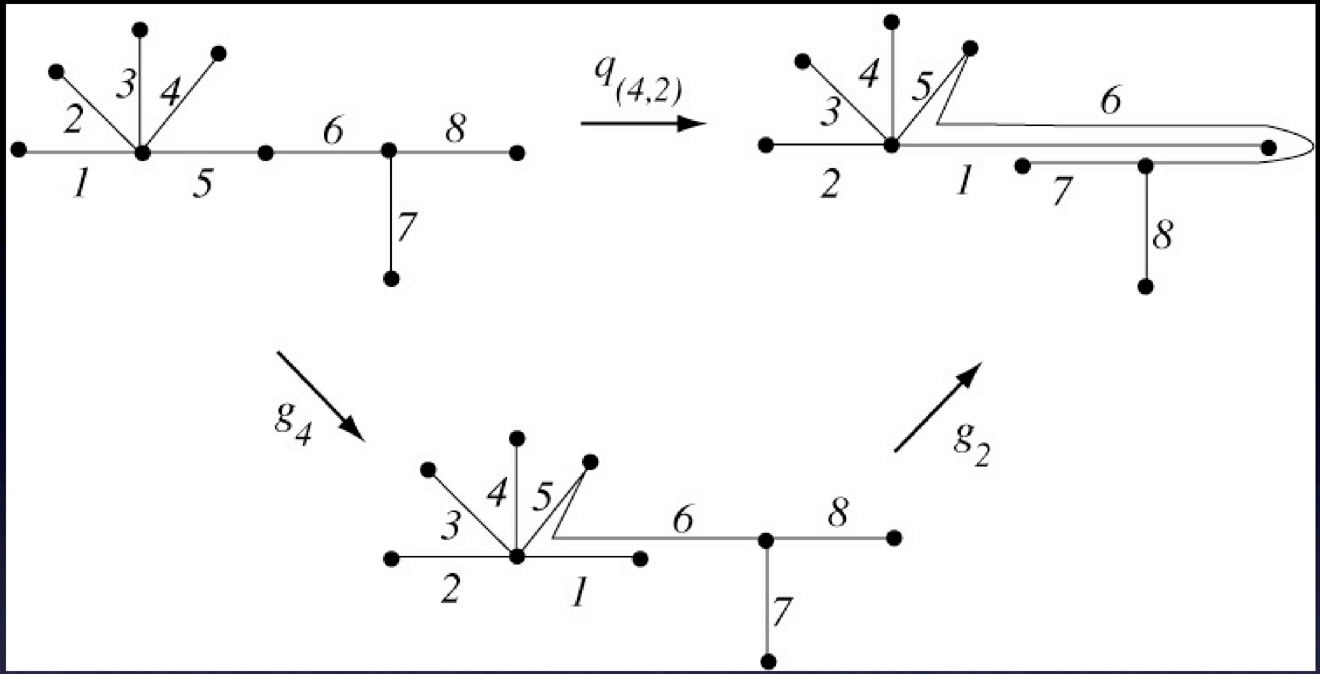
Take the extension $\widehat{g}$ of $g_{m_{k+1}} : \mathcal{G}_{m_{k+1}} \circlearrowleft$:

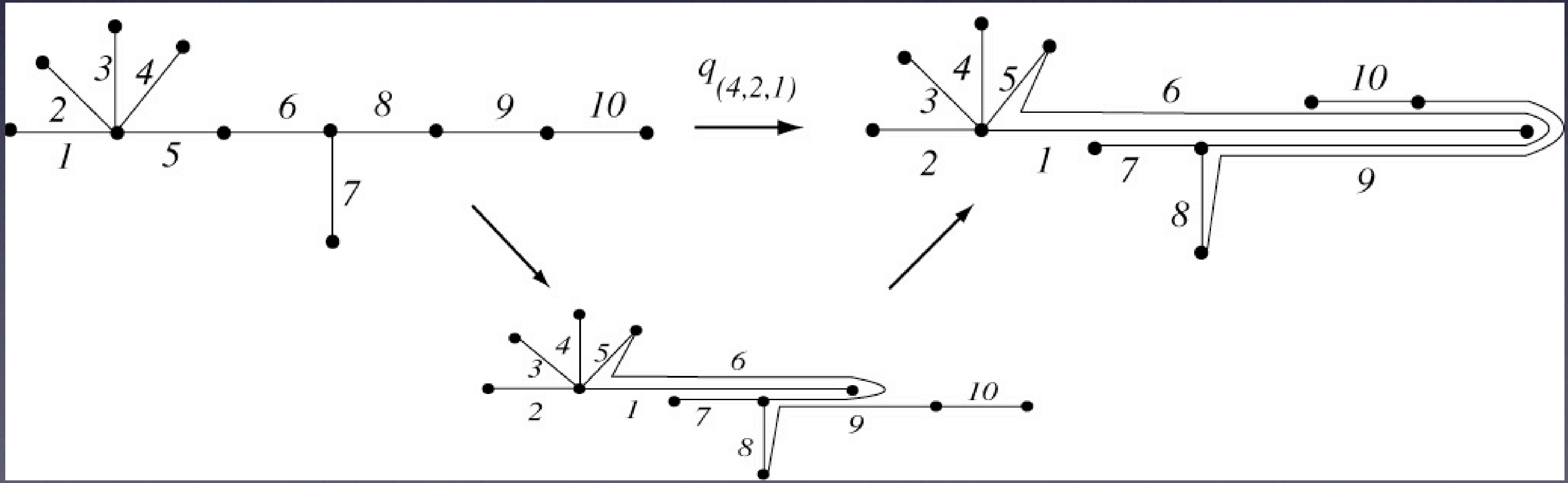
$$\widehat{g} : \mathcal{Q}_{(m_1,\cdots,m_{k+1})} \to \mathcal{Q}_{(m_1,\cdots,m_{k+1})}$$

Then we define

$$q_{(m_1,\cdots,m_{k+1})} := \widehat{g} \circ \widehat{q} : \mathcal{Q}_{(m_1,\cdots,m_{k+1})} \to \mathcal{Q}_{(m_1,\cdots,m_{k+1})}$$

(example)





Lem.

$$\lambda(M(q_{(m_1,\cdots,m_i,\cdots,m_{k+1})})) > \lambda(M(q_{(m_1,\cdots,m_i+1,\cdots,m_{k+1})})).$$

Cor. (Monotonicity)

$$\lambda(\beta_{(m_1,\cdots,m_i,\cdots,m_{k+1})}) > \lambda(\beta_{(m_1,\cdots,m_i+1,\cdots,m_{k+1})}).$$

proof of Thm. A (inductive formula)

Thm. A (Inductive formula) The dilatation of the pA braid $\beta_{(m_1,\cdots,m_{k+1})}$ is the largest root of the polynomial

$$t^{m_{k+1}} R_{(m_1,\cdots,m_k)}(t) + (-1)^{k+1} R_{(m_1,\cdots,m_k)_*}(t),$$

where $R_{(m_1,\cdots,m_i)}(t)$ is given inductively as follows:

$$\begin{aligned} R_{(m_1)}(t) &= t^{m_1+1}(t-1) - 2t, \text{ and} \\ R_{(m_1,\cdots,m_i)}(t) &= t^{m_i}(t-1)R_{(m_1,\cdots,m_{i-1})}(t) + (-1)^i 2t R_{(m_1,\cdots,m_{i-1})_*}(t) \text{ for } \end{aligned}$$

proof of Thm. A (inductive formula)

Fixing $m_1, \cdots, m_k \geq 1$, consider the family of combined tree map

$$\{q_{(m_{k+1})} = q_{(m_1,\cdots,m_{k+1})}\}_{m_{k+1}\geq 1}.$$

One can define the <u>dominant tree map</u>

$$\overline{r} = \overline{r}_{(m_1,\cdots,m_k)} : \mathcal{R} = \mathcal{R}_{(m_1,\cdots,m_k)} \to \mathcal{R}$$

whose transition matrix $M(\overline{r})$ equals the upper-left submatrix of $M(q_{(m_{k+1})})$:

$$(*) \qquad M(q_{(m_{k+1})}) = \left(\begin{array}{c|ccc} M(\overline{r}) & & & \\ & 1 & & \\ \hline & & \ddots & \\ & & & 1 \\ * \quad \cdots \quad * & & & \end{array}\right)$$

(How to define the *dominant tree map* $\overline{r} : \mathcal{R} \to \mathcal{R}$)

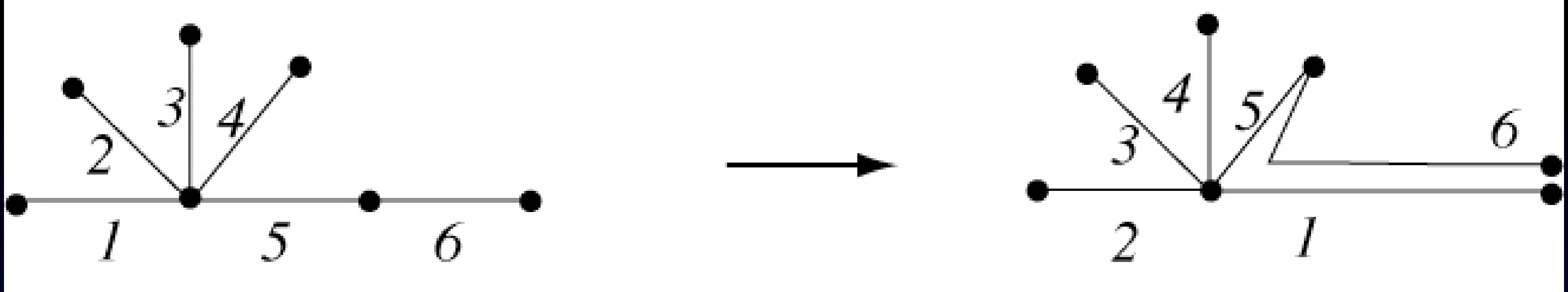
The tree $\mathcal{R}$ is obtained from the combined tree $\mathcal{Q}_{(m_1,\cdots,m_k)}$ together with the edge of $\mathcal{G}_{m_{k+1}}$.

For each $e \in E(\mathcal{R})$, the edge path $\overline{r}(e)$ is given by the edge path $q_{(m_{k+1})}(e)$ by eliminating edges which do not belong to the edge set $E(\mathcal{R})$.

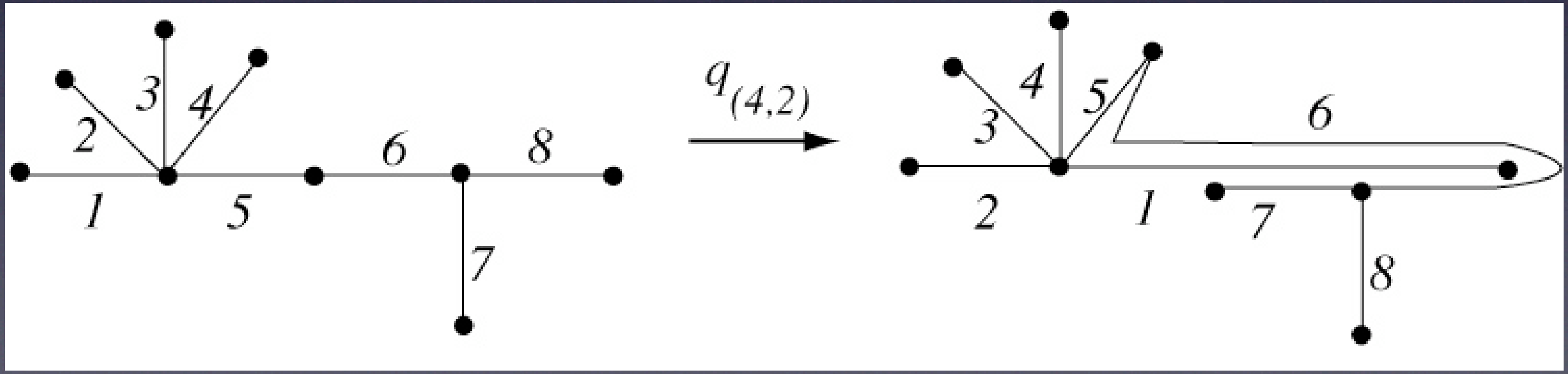
Note: The tree map $\overline{r}$ does not depend on the choice of m_{k+1}.

(example)

dominant tree map $\bar{r} : \mathcal{R} \to \mathcal{R}$ for $\{q_{(4,m)}\}_{m \geq 1}$.



(combined tree map $q_{(4,2)}$)



Lem. Let

$$R(t) = R_{(m_1,\cdots,m_k)}(t) = M(\overline{r})(t).$$

Then there exists a poly. $S(t)$ such that

$$M(q_{(m_1,\cdots,m_{k+1})})(t) = t^{m_{k+1}} R(t) + S(t)$$

(Proof.) Use this form

$$(*) \qquad M(q_{(m_1,\cdots,m_{k+1})}) = \left(\begin{array}{ccc|ccc} & M(\overline{r}) & & & & \\ & & & 1 & & \\ \hline & & & & \ddots & \\ & & & & & 1 \\ * & \cdots & * & & & \end{array}\right). \qquad \square$$

Mysterious Lem. (the poly. $R(t)$ knows the poly. $S(t)$.)

Let $S(t)$ be as in the previous lem. (i.e, $M(q_{(m_{k+1})})(t) = t^{m_{k+1}}R(t) + S(t)$.) Then

$$S(t) = (-1)^{k+1} R_*(t).$$

Thanks to the mysterious lemma, the proof of Thm. A is done:

$$\begin{aligned} & \lambda(\beta_{(m_1,\cdots,m_{k+1})}) \qquad \text{(the dilatation of the braid)} \\ = \ & \lambda(M(q_{(m_1,\cdots,m_{k+1})})) \qquad \text{(the spectral radius of the matirx)} \\ = \ & \text{the largest root of } t^{m_{k+1}} R_{(m_1,\cdots,m_k)}(t) + (-1)^{k+1} R_{(m_1,\cdots,m_k)_*}(t) \quad \square \end{aligned}$$

Thm B.(Asymptotic behavior)

$$\lim_{m_1,\cdots,m_{k+1}\to\infty} \lambda(\beta_{(m_1,\cdots,m_{k+1})}) = 1$$

Final Lemma.

$$\lim_{m_1\to\infty}\lim_{m_2\to\infty}\cdots\lim_{m_{k+1}\to\infty} \lambda(\beta_{(m_1,\cdots,m_{k+1})}) = 1$$

Once we prove the lemma, then our task is done, since the monotonicity of the dilatation holds:

if $m'_i \geq m_i$ for each i, then $\lambda(\beta_{(m_1,\cdots,m_{k+1})}) \geq \lambda(\beta_{(m'_1,\cdots,m'_{k+1})})$.

(Proof of the final lemma.)

For an integral poly. $Q(t)$, let $\lambda(Q(t))$ be the largest absolute value of roots of $Q(t)$.

Let us consider a family of polynomials

$$Q_n(t) = t^n R(t) \pm S(t),$$

where $R(t)$ is a monic integral poly. and $S(t)$ is an integral poly.

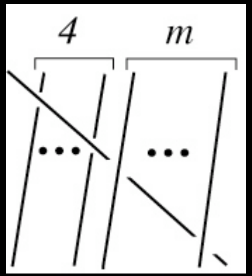
(catch phrase)
"The roots of $R(t)$ dominate those of $Q_n(t)$ asymptotically"
(dominate: 支配する)

Key lemma. Suppose that $R(t)$ has a root outside the unit circle. Then,

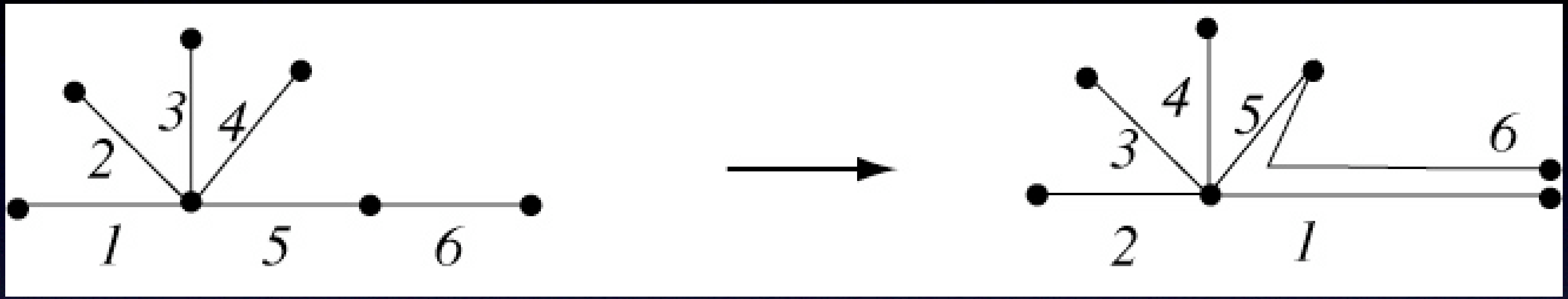
$$\lambda(R(t)) = \lim_{n\to\infty} \lambda(Q_n(t)).$$

Key lemma together with the inductive formula [Thm A] to compute $R_{(m_1,\cdots,m_i)}(t)$ gives us the proof of the final lemma. $\square$

Example. (recipe to compute $\lambda(\beta_{4,m})$ for $m \geq 1$)



1. Consider the dominant tree map $\overline{r} = \overline{r}(4)$ for $\{q_{(4,m)}\}$.



2. Compute the characteristic poly. of $M(\overline{r})(t) = t^6 - t^5 - 2t$.

3. Compute its reciprocal $(t^6 - t^5 - 2t)_* = -2t^5 - t + 1$

$\Longrightarrow$ For each $m \geq 1$, the dilatation $\lambda(\beta_{4,m})$ is the largest root of

$$t^m(t^6 - t^5 - 2t) + (-2t^5 - t + 1).$$