Nielsen number in a graph — a road map?

Boju Jiang

Peking University, Beijing, China

East Asian School of Knots and Related Topics
Tokyo University, Jan 22, 2008

Abstract

PROJECT GOAL: Find algorithmic solutions to the following:

Question 1. The computation of Nielsen number for self-maps of finite graphs (= 1-dimensional cell complexes).

Question 2. The twisted conjugacy problem in free groups of finite rank.

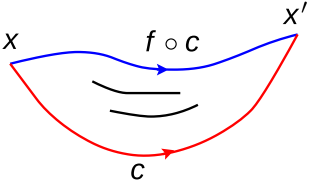
This is a preliminary report. Comments are most welcome.

Outline

Outline

1. Nielsen fixed point theory
   - Fixed point class
   - Coordinate of fixed point class
2. Twisted conjugacy in free groups
   - Twisted conjugacy generalities
   - Free groups via graph maps
3. The Bestvina-Handel theory of train tracks
   - A kind of normal form
   - Some terminology
   - Relative train tracks
4. The fixed subgroup in free groups
   - Computing the fixed subgroup

Fixed point class



Nielsen path (color code: f sends red to blue)

Let X be a connected compact polyhedron, and $f : X \to X$ a self-map.

Definition

Two fixed points $x, x' \in \mathrm{Fix}\, f$ are in the same fixed point class $\iff$ there is a path c (called a Nielsen path) from x to x' such that $c \simeq f \circ c$ rel endpoints.

Definition

A fixed point class $\boldsymbol{F}$ is essential if its index is nonzero, i.e., $\mathrm{ind}(\boldsymbol{F}, f) \neq 0$. The number of essential fixed point classes is called the Nielsen number of f, denoted $N(f)$.

Invariance

Theorem (Homotopy invariance)

If $f_0 \simeq f_1 : X \to X$, then $N(f_0) = N(f_1)$.

Theorem (Commutation invariance)

Suppose $\phi : X \to Y$ and $\psi : Y \to X$. Then $N(\psi \circ \phi) = N(\phi \circ \psi)$.

A sequence $\{f_i : X_i \to X_i \mid i = 0, \ldots, k\}$ of self-maps is a mutation if for each i, either

1. $X_{i+1} = X_i$ and $f_i \simeq f_{i+1}$, or
2. f_{i+1} is obtained from f_i by commutation.

Corollary (Mutation invariance)

Nielsen number is a mutation invariant.

Invariance

Theorem (Homotopy invariance)

If $f_0 \simeq f_1 : X \to X$, then $N(f_0) = N(f_1)$.

Theorem (Commutation invariance)

Suppose $\phi : X \to Y$ and $\psi : Y \to X$. Then $N(\psi \circ \phi) = N(\phi \circ \psi)$.

A sequence $\{f_i : X_i \to X_i \mid i = 0, \ldots, k\}$ of self-maps is a mutation if for each i, either

1. $X_{i+1} = X_i$ and $f_i \simeq f_{i+1}$, or
2. f_{i+1} is obtained from f_i by commutation.

Corollary (Mutation invariance)

Nielsen number is a mutation invariant.

Minimality

Theorem (Homotopy lower bound)

$N(f)$ is a lower bound for the number of fixed points in the homotopy class of f.

$$N(f) \leq \mathrm{MF}[f] := \min\{\# \operatorname{Fix} g \mid g \simeq f : X \to X\}.$$

Theorem (Minimality)

Suppose X has no local cut points. Then $N(f) = \mathrm{MF}[f]$ for every map $f : X \to X$ if and only if X is <u>not</u> a surface with negative Euler characteristic.

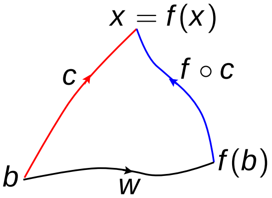
In contrast to the theoretical beauty, the Nielsen number is difficult to compute, even if X is 1-dimensional.

A recent survey:

E. L. Hart, Algebraic techniques for calculating the Nielsen number on hyperbolic surfaces.
pp.463–487 in Handbook of Topological Fixed Point Theory, (Brown et al. eds.), Springer, 2005.

Outline

Coordinate of a fixed point



Coordinate of fixed point

Let b be the base point of X. Denote $\pi := \pi_1(X, b)$. Choose a path w from b to $f(b)$ as the reference path. Define the endomorphism $f_\pi : \pi \to \pi$ by

$$f_\pi(\langle a \rangle) := \langle w(f \circ a)w^{-1} \rangle \qquad \text{for all } \langle a \rangle \in \pi.$$

Pick a path c from the base point b to a fixed point x. We intend to label x with the loop class

$$\langle w(f \circ c)c^{-1} \rangle \quad \in \pi,$$

up to the ambiguity arising from the choice of c.

Coordinate of a fixed point

Definition

Suppose $\phi : G \to G$ is a group endomorphism. Elements $g, g' \in G$ are ϕ-conjugate, written $g \sim_\phi g'$ or $[g]_\phi = [g']_\phi$, if $\exists h \in G$ such that $g' = (h\phi)gh^{-1}$.

The set of ϕ-conjugacy classes in G is denoted $\mathcal{R}(\phi)$, called the Reidemeister set of ϕ.

Definition

Pick an arbitrary path c from the base point b to a fixed point x. The coordinate of x is the f_π-conjugacy class

$$\mathrm{cd}_\pi(x, f) = [\langle w(f \circ c)c^{-1} \rangle]_{f_\pi} \qquad \in \mathcal{R}(f_\pi).$$

Coordinate of fixed point class

Proposition

Two fixed points $x, x' \in \operatorname{Fix} f$ are in the same fixed point class if and only if they have the same coordinate.

Definition

The coordinate of a fixed point class $\boldsymbol{F}$ is the common coordinate of all its members. For any path c from b to $\boldsymbol{F}$,

$$\mathrm{cd}_\pi(\boldsymbol{F}, f) = [\langle w(f \circ c)c^{-1}\rangle]_{f_\pi} \qquad \in \mathcal{R}(f_\pi).$$

Coordinate of fixed point class

Proposition

Two fixed points $x, x' \in \mathrm{Fix}\, f$ are in the same fixed point class if and only if they have the same coordinate.

Definition

The coordinate of a fixed point class $\boldsymbol{F}$ is the common coordinate of all its members. For any path c from b to $\boldsymbol{F}$,

$$\mathrm{cd}_\pi(\boldsymbol{F}, f) = [\langle w(f \circ c)c^{-1} \rangle]_{f_\pi} \qquad \in \mathcal{R}(f_\pi).$$

Twisted Lefschetz invariant

Definition

The twisted Lefschetz invariant of f is the formal sum

$$L_\pi(f) := \sum_{x \in \mathrm{Fix} f} \mathrm{ind}(x, f) \cdot \mathrm{cd}_\pi(x, f)$$
$$= \sum_{\boldsymbol{F}} \mathrm{ind}(\boldsymbol{F}, f) \cdot \mathrm{cd}_\pi(\boldsymbol{F}, f) \quad \in \boldsymbol{Z}\mathcal{R}(f_\pi).$$

The twisted Lefschetz invariant also enjoys the mutation invariance.

The key to the computation of $N(f)$ and $L_\pi(f)$ is the twisted conjugacy problem in π.

Outline

Commutation

Assumption: $\phi : G \to G$, $g \mapsto g\phi$ is a group endomorphism.

Note: Homomorphisms act on the right. The composition of homomorphisms is from left to right.

Recall the ϕ-conjugacy problem: Given $g, g' \in G$, to decide whether they are ϕ-conjugate, i.e., whether $\exists h \in G$ such that $g' = \phi(h)gh^{-1}$.

Proposition (Commutation)

Suppose $\alpha : G \to G'$ and $\beta : G' \to G$ are homomorphisms. Then the $\alpha\beta$-conjugacy in G is decidable $\iff$ the $\beta\alpha$-conjugacy in G' is decidable.

HNN extension

Proposition (HNN EXTENSION)

Consider the HNN extension

$$G*_\phi = \langle G, t \mid t^{-1}gt = x\phi, \forall g \in G \rangle.$$

Then g and g' in G are ϕ-conjugate if and only if gt and $g't$ are conjugate in $G*_\phi$.

Thus, the ϕ-conjugacy in G is decidable if the ordinary conjugacy in $G*_\phi$ is decidable.

B. Jiang, A characterization of fixed point classes.
Contemporary Math. vol.72 (1988) 157–160.

Free extention

Lemma

Let $g, g' \in G$. Then $g \sim_\phi g' \iff g^{-1}g' \sim_{\phi^g} 1$, where the endomorphism ϕ^g is defined by $h\phi^g = g^{-1}(h\phi)g$, $\forall h \in G$.

Lemma (Free extention)

Assume that $g \in G$. Extend ϕ to $\phi' : G * \langle z \rangle \to G * \langle z \rangle$ by letting $z\phi' = gzg^{-1}$. Then, $g \sim_\phi 1 \iff \operatorname{Fix} \phi' \neq \operatorname{Fix} \phi$.

Note that obviously $\operatorname{Fix} \phi = G \cap \operatorname{Fix} \phi'$.

O. Bogopolski, A. Martino, O. Maslakova, E. Ventura, The conjugacy problem is solvable in free-by-cyclic groups.
Bull. London Math. Soc. **38** (2006) 787–794.

Outline

topological representative

Let X be a connected finite graph (= 1-dimensional cell complex) such that $\pi_1(X) = F_n$ (= the free group of rank n).

Cellular maps between graphs are always assumed to be tight, i.e., locally injective (or locally constant) in the interior of each edge.

Definition

A cellular map $f : X \to X$ is a topological representative of an endomorphism $\phi : F_n \to F_n$, if $\phi = f_* : F_n \to F_n$ with respect to some base point and reference path.

Reduction to π_1-injective

Lemma

Let $f : X \to X$ be a self-map of a connected graph. Then f is a mutant of a graph map $f' : X' \to X'$ such that f' is π_1-injective and $\operatorname{rank} \pi_1(X') \leq \operatorname{rank} \pi_1(X)$.

B. Jiang, Bounds for fixed points on surfaces.
Math. Ann. **311** (1998) 467–479.

The proof is algorithmic, using the folding move of Stallings.

Hence, in the ϕ-conjugacy problem for free groups, we can focus on injective ϕ.

Outline

The Bestvina-Handel theory of train tracks

Given a free group automorphism $\phi : F_n \to F_n$, Bestvina-Handel gave a sort of normal form (via mutation), analogous to Thurston's normal form for surface homeomorphisms (via isotopy).

This normal form, called a "(relative) train track map", consists of a cellular map on a graph and has good dynamics. It is used to prove the Scott conjecture that the rank of Fix ϕ is at most n.

M. Bestvina, M. Handel, Train tracks and automorphisms of free groups.
Ann. of Math. **135** (1992) 1–51.

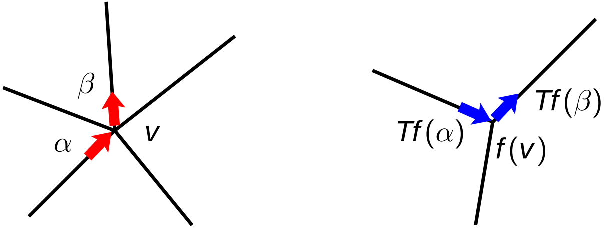
It turns out that this theory works also for injective endomorphisms.

Outline

Some terminology

A cellular map $f : G \to G$ that is locally injective on edges induces a tangent map Tf on the “germs at vertices” (or “tips of edges”).

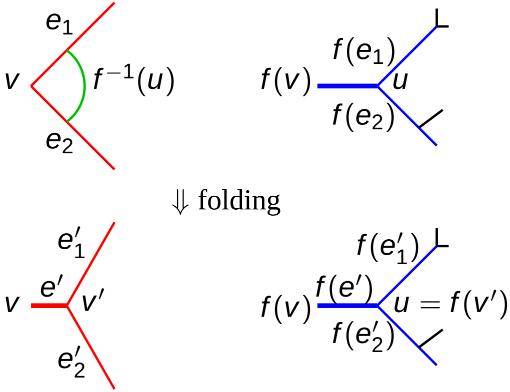
A turn is a pair of germs at a vertex, one incoming and one outgoing.



Turn and tangent map (Tf : red $\mapsto$ blue)

Some terminology

The folding move of Stallings.



Folding a degenerate turn

Some terminology

A turn is legal if it never degenerates under iterates of Tf. Otherwise it is illegal.

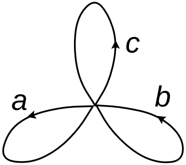
A map $f : G \to G$ is a train track map if on the interior of each edge e, the iterates f^k are all locally injective. In other words, $f(e)$ does not contain any illegal turns.

An indivisible Nielsen path (= INP) is a Nielsen path of f that is not a nontrivial concatenation of two Nielsen paths.

The INPs generate a finite graph that is associated to the fixed subgroups in π.

Some terminology

The transition matrix $M = (m_{ij})$ of f has m_{ij} equal to the number of times that the f-image of j-th edge crosses i-th edge.



$$f_\pi : \begin{cases} a \mapsto b \\ b \mapsto c \\ c \mapsto b^{-1}ab \end{cases} \qquad M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

Transition matrix

If M is irreducible, there is a unique metric on G (normalized so that the total length of edges is 1) such that f expands lengths of edges by a stretching factor $\lambda \geq 1$. When $\lambda = 1$, f cyclically permutes the edges of G.

Outline

Relative train track map

A cellular map $f : G \to G$ that is locally injective on edges is a relative train track map (RTT) if there is a filtration

$$\emptyset = G_0 \subset \cdots \subset G_m = G$$

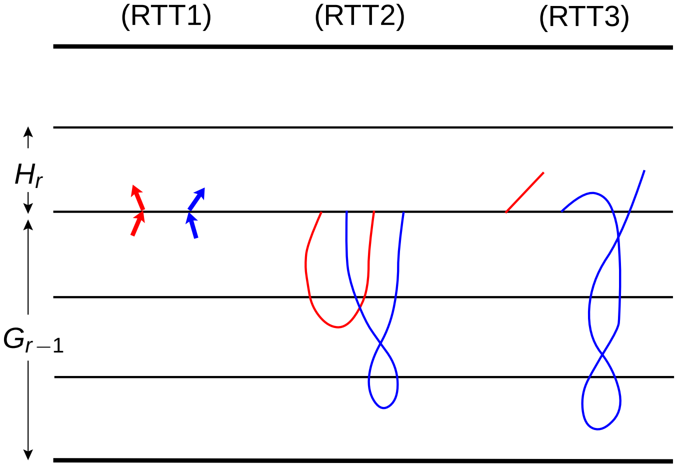
into f-invariant subgraphs with the following properties. Denote by H_r (called a stratum) the closure of $G_r \setminus G_{r-1}$, and by M_r the part of the transition matrix corresponding to H_r. Then M_r is the zero matrix or an irreducible matrix. If M_r is irreducible then:

(RTT1) the tangent map Tf sends germs in H_r to germs in H_r,

(RTT2) if α is a nontrivial path in G_{r-1} with endpoints in $H_r \cap G_{r-1}$ then $f(\alpha)$ is also a nontrivial path with endpoints in $H_r \cap G_{r-1}$, and

(RTT3) every edge in H_r is mapped to a path that does not cross illegal turns in H_r.

Relative train track map



Relative train track map

Relative train track maps

Theorem

Every injective endomorphism $\phi : F_n \to F_n$ admits a relative train track representative.

Furthermore, it can be made stable so that for each r, at most one INP in G_r can intersect the interior of H_r.

The proof is constructive and algorithmic.

W. Dicks, E. Ventura, The group fixed by a family of injective endomorphisms of a free group.
Contemporary Math. **195**, Amer. Math. Soc. 1996.

Theorem (The Scott conjecture)

The rank of $\mathrm{Fix}(\phi : F_n \to F_n)$ *is at most* n.

Outline

Computing the fixed subgroup

The train track theory is for outer automorphisms of free groups. But the fixed subgroup Fix ϕ is sensitive to composition with inner automorphisms.

When ϕ is an automorphism, there is an algorithm to find a basis of Fix ϕ.

O. S. Maslakova, The fixed point group of a free group automorphism.
Algebra Logic **42** (2003), 237–265.

First modify the Bestvina-Handel RTT into a based version for better control of the base point and reference path. Then find the INPs.

Perhaps we can do similarly for injective endomorphisms.

Summary

- The Nielsen number of a graph self-map is computable. The Dicks-Ventura induction procedure may already imply an algorithm for π_1-injective self-maps.
- The twisted conjugacy problem for endomorphisms of free groups looks accessible.
- Outer automorphisms of free groups have been much studied in geometric group theory. What about outer endomorphisms?