Configuration space integral and Poisson structure on the homology of the space of framed long knots

Keiichi Sakai

1 Long knots

Definition

A **long knot** f in $\mathbb{R}^n$;

$$f : \mathbb{R}^1 \hookrightarrow \mathbb{R}^n,$$
$$f(t) = (0, \dots, 0, t) \quad \text{if } |t| \geq 1.$$

A **framed long knot** g in $\mathbb{R}^n$;

$$g : B^{n-1} \times \mathbb{R}^1 \hookrightarrow B^{n-1} \times \mathbb{R}^1,$$
$$g(x, t) = (x, t) \quad \text{if } |t| \geq 1$$

(B^{n-1} : the unit ball in $\mathbb{R}^{n-1}$).

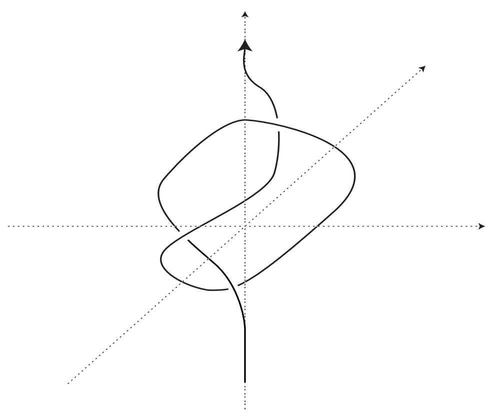

Figure 1: A long knot

Defintion

$\mathcal{K}_n$ ($\widetilde{\mathcal{K}}_n$) : the space of (framed) long knots, equipped with C^∞-topology.

$\mathcal{K}_3$ has been studied in detail by R. Budney [2], so we will study the case $n > 3$.

Problem

Compute $H_*(\mathcal{K}_n)$, $H_*(\widetilde{\mathcal{K}}_n)$ when $\underline{n > 3}$.

One construction : for a **chord diagram** Γ with k chords, define

$$\alpha(\Gamma) : (S^{n-3})^k \longrightarrow \widetilde{\mathcal{K}}_n$$

as in Figure 2 ; then we have an injection of graded algebras

$$\alpha : \mathcal{A} \hookrightarrow \bigoplus_{k \geq 0} H_{(n-3)k}(\widetilde{\mathcal{K}}_n)$$

$\mathcal{A}$: the algebra of chord diagrams modulo "4-term relations."

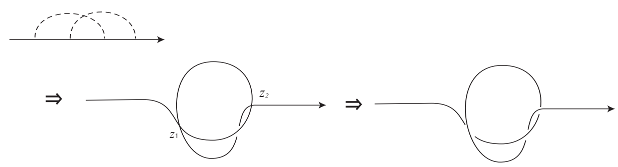

Figure 2: Blow-up of self-intersections

Problem

Find any other classes, possibly coming from "non-trivalent" diagrams.

2 Poisson structure

The connecting sum induces a product

$$* : H_p(\widetilde{\mathcal{K}}_n) \otimes H_q(\widetilde{\mathcal{K}}_n) \to H_{p+q}(\widetilde{\mathcal{K}}_n).$$

More structure ;

Poisson structure

A degree-one Lie bracket

$$\lambda : H_p(\widetilde{\mathcal{K}}_n) \otimes H_q(\widetilde{\mathcal{K}}_n) \to H_{p+q+1}(\widetilde{\mathcal{K}}_n)$$

satisfying the Leibniz rule

$$\lambda(x, y * z) = \lambda(x, y) * z \pm y * \lambda(x, z).$$

We call λ the **Browder operation** [4].

An **action of little disks operad** on $\widetilde{\mathcal{K}}_n$ induces λ :

Action of little disks operad [2]

λ : induced by the map (Figure 3)

$$\kappa(2) : \mathcal{O}_2(2) \times (\widetilde{\mathcal{K}}_n)^2 \longrightarrow \widetilde{\mathcal{K}}_n$$

$\mathcal{O}_2 = \{\mathcal{O}_2(k)\}$: **little disks operad**,

$$\mathcal{O}_2(k) = \left\{k \text{ disjoint disks in } D^2\right\}.$$

In fact $\mathcal{O}_2$ "acts" on $\widetilde{\mathcal{K}}_n$ [2] ;

$$\kappa(k) : \mathcal{O}_2(k) \times (\widetilde{\mathcal{K}}_n)^k \to \widetilde{\mathcal{K}}_n, \quad k \geq 1.$$

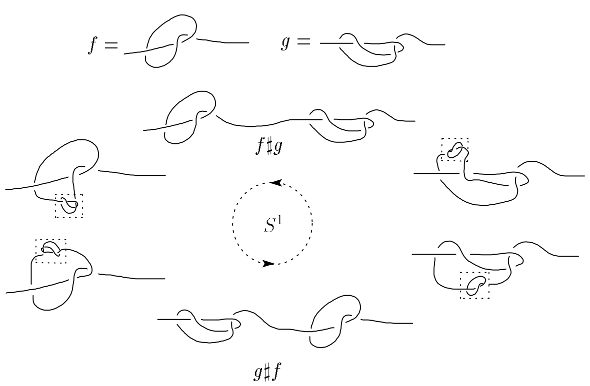

Figure 3: The map $\kappa(2)$

Remark

$\mathcal{O}_2(k) \simeq \mathrm{Conf}_k(\mathbb{R}^2)$: the **configuration space**. In particular $\mathcal{O}_2(2) \simeq S^1$.

Consider $\lambda(e, v_2) \in H_{3(n-3)+1}(\widetilde{\mathcal{K}}_n)$ where

$$e = \alpha(\Gamma_0) \in H_{n-3}(\widetilde{\mathcal{K}}_n),$$
$$v_2 = \alpha(\Gamma_1) \in H_{2(n-3)}(\widetilde{\mathcal{K}}_n)$$

for chord diagrams in Figure 4.

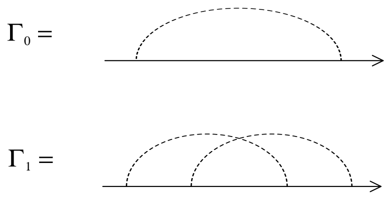

Figure 4: Chord diagrams Γ_0, Γ_1

Main Theorem ([6])

If $n > 3$ is odd, then

$$\lambda(e, v_2) \in H_{3(n-3)+1}(\widetilde{\mathcal{K}}_n)$$

is not zero.
$\lambda(e, v_2) \notin \alpha(\mathcal{A})$ because of its degree.

Proof. Construction of a dual cocycle via the **configuration space integral**.

3 Configuration space integrals

A. Cattaneo, P. Cotta-Ramusino and R. Longoni [3] proved the following.

Theorem

When $n > 3$, there is a cochain map

$$I : \mathcal{D}^{k,l} \longrightarrow \Omega_{DR}^{(n-3)k+l}(\mathcal{K}_n)$$

(see Example below; $\mathcal{D}^{**}$ is a certain **graph complex**), inducing an injection

$$I : H^{k,0}(\mathcal{D}^{**}) \hookrightarrow H_{DR}^{(n-3)k}(\mathcal{K}_n).$$

In fact

$$H^{k,0}(\mathcal{D}^{**}) \cong (\mathcal{A}_k / \text{"1-term relations"})^*. \quad \square$$

Remark

- $\forall$ vertices of graphs : of valence ≥ 3.
- A graph $\Gamma \in \mathcal{D}^{k,l} \Rightarrow \chi(\Gamma) = 1 - k$, and

$$\Gamma : \text{trivalent} \iff l = 0.$$

In fact $I(H^{k,0}(\mathcal{D}^*))$ is "dual" to $\alpha(\mathcal{A})$.

Example

$I(\Gamma_2) \in \Omega^{2(n-3)}_{DR}(\mathcal{K}_n)$ for Γ_2 in Figure 5.

$$C_{\Gamma_2} := \left\{ \begin{array}{c} (f;(x_i)_{i=1}^4) \in \\ \mathcal{K}_n \times \mathrm{Conf}_4(\mathbb{R}^n) \end{array} \middle| \begin{array}{l} \text{for } i=1,2,3, \\ x_i = f(t_i) \text{ for} \\ \exists\, t_1 < t_2 < t_3 \end{array} \right\}$$

For each edge $\overrightarrow{i4}$ $(1 \le i \le 3)$, define

$$\varphi_{i4} : C_{\Gamma_2} \to S^{n-1}, \quad \varphi_{i4}(f,x) := \frac{x_4 - x_i}{|x_4 - x_i|}.$$

Also define

$$\theta_{\Gamma_2} := \bigwedge_{1 \le i \le 3} \varphi_{i4}^* vol_{S^{n-1}} \in \Omega^{3(n-1)}_{DR}(C_{\Gamma_2}).$$

Integrate θ_{Γ_2} along the fiber of the fibration

$$p : C_{\Gamma_2} \longrightarrow \mathcal{K}_n, \quad p(f,x) = f,$$

we have a differential form

$$I(\Gamma_2) := p_* \theta_{\Gamma_2} \in \Omega^{2(n-3)}_{DR}(\mathcal{K}_n). \quad \square$$

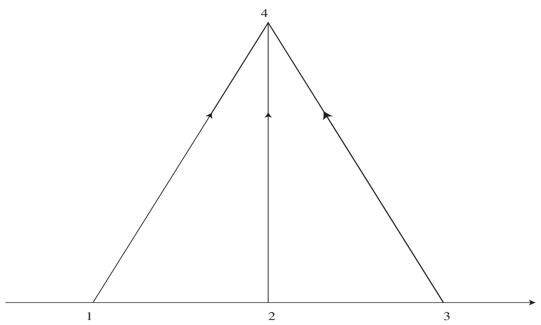

Figure 5: The graph $\Gamma_2 \in \mathcal{D}^{2,0}$

The differential of $\mathcal{D}$: the contractions of "edges" and "arcs" ($\overrightarrow{14}$, $\overrightarrow{12}$ etc).

Remark

The map I is a generalization of integral expressions of Vassiliev invariants for knots in $\mathbb{R}^3$ [1, 5]. $\square$

$H^{k,l}(\mathcal{D}^{**})$, $l > 0$ (the spaces of <u>non-trivalent</u> graph cocycle) have been unknown, but :

Proposition ([6])

If $n > 3$ is odd, then $H^{3,1}(\mathcal{D}^{**}) \cong \mathbb{R}$. $\square$

A generator $\Gamma \in H^{3,1}(\mathcal{D}^{**})$ gives

$$I(\Gamma) \in H^{3(n-3)+1}_{DR}(\mathcal{K}_n).$$

Theorem ([6])

If $n > 3$ is odd, then the Kronecker pairing

$$\langle I(\Gamma), r_* \lambda(e, v_2) \rangle \neq 0,$$

where $r : \widetilde{\mathcal{K}}_n \to \mathcal{K}_n$ is the forgetful map. $\square$

Thus both $\lambda(e, v_2)$ and $I(\Gamma)$ are non-trivial.

References

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Graduate School of Mathematical Sciences,
University of Tokyo
ksakai@ms.u-tokyo.ac.jp