Isometries on SU(2)-representation spaces of knot groups and twisted Alexander functions

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We study two sorts of actions on the space of conjugacy classes of irreducible $SU(2)$-representations of a knot group and describe a symmetry of the twisted Alexander function on the space.

Notation and Facts.

$K \subset S^3$: a tame knot,
$E_K := S^3 \setminus N(K)$; $N(K)$: an open tubular neighborhood,
$G_K := \pi_1 E_K$, $\mu, \lambda \in G_K$: a meridian-longitude pair,
$\alpha\colon G_K \to \langle t \rangle$: an abelianization map,
$\mathcal{R}(K)$: the space of conjugacy classes of irreducible $SU(2)$-representations,
$\mathcal{S}(K)$: the set of conjugacy classes of irreducible metabelian $SU(2)$-representations.

- $\mathrm{Hom}(G_K, SU(2))$ is a topological space via the compact open topology where G_K carries the discrete topology.
- We call $\rho \in \mathrm{Hom}(G_K, SU(2))$ a *metabelian representation* if $\rho([G_K, G_K])$ is abelian. From the result of Lin [L], we have

$$\sharp \mathcal{S}(K) = \frac{1}{2}(|\Delta_K(-1)| - 1), \qquad (\star)$$

  where $\Delta_K(t)$ is the Alexander polynomial of K.

Definition 0.1.
An irreducible representation $\rho \in \mathrm{Hom}(G_K, SU(2))$ is called *regular* if $\dim H^1(E_K; \mathfrak{su}(2)_{\mathrm{Ad} \circ \rho}) = 1$. We denote by $\mathrm{Reg}(K) \subset \mathcal{R}(K)$ the space of conjugacy classes of regular representations.

- $\mathrm{Reg}(K)$ is an open 1-dimensional manifold in $\mathcal{R}(K)$ (by Heusener and Klassen [HK]).
- Dubois constructed a canonical volume form τ_K on $\mathrm{Reg}(K)$ via non-acyclic Reidemeister torsion in [D], which induces a canonical orientation and Riemannian metric on $\mathrm{Reg}(K)$. Explicitly, we define an inner product $g\colon T\,\mathrm{Reg}(K) \otimes T\,\mathrm{Reg}(K) \to \mathbb{R}$ by $g(v, w) := \tau_K(v)\tau_K(w)$, where $(v, w) \in T\,\mathrm{Reg}(K) \otimes T\,\mathrm{Reg}(K)$.

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1 (-1)-involution

There is an involution on $\mathrm{Hom}(G_K, SU(2))$ defined by

$$\rho \;\mapsto\; (\gamma \mapsto \alpha(\gamma)|_{t=-1}\rho(\gamma)),$$

where $\gamma \in G_K$, which naturally induces an involution ι on $\mathcal{R}(K)$ and $\mathrm{Reg}(K)$.

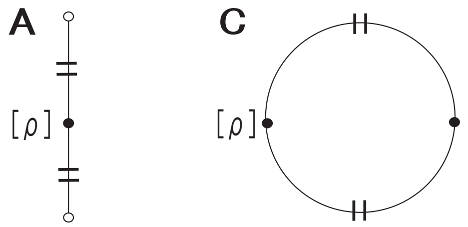
Theorem 1.1.

$\iota\colon\ \mathrm{Reg}(K) \to \mathrm{Reg}(K)$ *is an orientation reversing isometry.*

From this theorem and the fact proved by Nagasato and Yamaguchi [NY] that $[\rho] \in \mathcal{R}(K)$ is a fixed point of ι if and only if $[\rho] \in \mathcal{S}(K)$, we can detect the distribution of $\mathcal{S}(K)$ in $\mathrm{Reg}(K)$.

Theorem 1.2.

(i)If $[\rho] \in \mathcal{S}(K)$ *is a point on an arc component* A *of* $\mathrm{Reg}(K)$*,* $[\rho]$ *is a unique point on* $\mathcal{S}(K) \cap A$ *and* $[\rho]$ *is at the center of* A*.*
(ii)If $[\rho] \in \mathcal{S}(K)$ *is a point on a circle component* C *of* $\mathrm{Reg}(K)$*, there are exactly two points on* $\mathcal{S}(K) \cap C$ *and they are antipodal to each other.*



Moreover, from this theorem and (☆), we obtain the following corollary.

Corollary 1.3.

If $\sharp\mathcal{S}(K) \cap \mathrm{Reg}(K)$ *is odd, then there is an arc component* $A \subset \mathrm{Reg}(K)$*. In particular, if* $\mathcal{S}(K) \subset \mathrm{Reg}(K)$ *and* $|\Delta_K(-1)| \equiv -1 \mod 4$*, then there is an arc component* $A \subset \mathrm{Reg}(K)$*.*

Since all the metabelian representations are explicitly computable, $\sharp\mathcal{S}(K) \cap \mathrm{Reg}(K)$ is also computable for any knot K.

2 Actions by automorphism groups of knot groups

The automorphism group $\mathrm{Aut}(G_K)$ acts on the representation space $\mathrm{Hom}(G_K, SU(2))$ via pullback, namely we define

$$\varphi^*\rho := \rho \circ \varphi,$$

where $\varphi \in \mathrm{Aut}(G_K)$. It can be checked that this action naturally induces the action on $\mathcal{R}(K)$ and $\mathrm{Reg}(K)$ by $\mathrm{Out}(G_K)$, $\mathrm{Out}_p(G_K)$ and $\mathrm{Out}_p^{\pm}(G_K)$, where

$$\mathrm{Out}_p(G_K) := \{[\varphi] \in \mathrm{Out}(G_K) \mid \varphi(\langle \mu, \lambda \rangle)) = \langle \mu, \lambda \rangle\},$$
$$\mathrm{Out}_p^{\pm}(G_K) := \{[\varphi] \in \mathrm{Out}(G_K) \mid \varphi(\langle \mu, \lambda \rangle)) = \langle \mu, \lambda \rangle, \det \varphi|_{\langle \mu, \lambda \rangle} = \pm 1\}.$$

Theorem 2.1.

(i)For any class $[\varphi] \in \mathrm{Out}_p(G_K)$, $[\varphi]^* : \mathrm{Reg}(K) \to \mathrm{Reg}(K)$ is an isometry.
(ii)For any class $[\varphi] \in \mathrm{Out}_p^{+}(G_K)$ with finite order, $[\varphi]^*\colon \mathrm{Reg}(K) \to \mathrm{Reg}(K)$ preserves the orientation.
(iii)For any class $[\varphi] \in \mathrm{Out}_p^{+}(G_K)$ (resp. $\mathrm{Out}_p^{-}(G_K)$) and a component D which have a class $[\rho]$ such that $\rho(\lambda) = 1$, $[\varphi]^*|_D$ preserves (resp. reverses) the orientation.

From this theorem, we can obtain a following necessary condition for a knot to be amphicheiral.

Theorem 2.2.

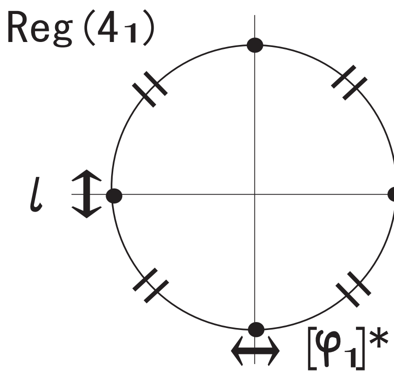
Let K be an amphicheiral knot. If $\sharp\{[\rho] \in \mathrm{Reg}(K) \mid \mathrm{tr}\,\rho(\mu) = 0\}$ is finite, then $\sharp\mathcal{S}(K) \cap \mathrm{Reg}(K)$ is even. Moreover, if $\mathcal{S}(K) \subset \mathrm{Reg}(K)$, then $|\Delta_K(-1)| \equiv 1 \mod 4$.

For instance, if K is small, namely E_K contains no essential surface, then $\sharp\{[\rho] \in \mathrm{Reg}(K) \mid \mathrm{tr}\,\rho(\mu) = 0\}$ is finite.

Example 2.3. $\underline{K = \text{the figure eight knot } 4_1}$.
Direct computation gives $\mathrm{Reg}(K) = \mathcal{R}(K) \approx S^1$, $\sharp\mathcal{S}(K) = 2$ and

$$\mathrm{Out}(G_K) = \langle\ [\varphi_1], [\varphi_2] \ \mid\ [\varphi_1]^4 = 1, [\varphi_2]^2 = 1, [\varphi_2][\varphi_1][\varphi_2] = [\varphi_1]^{-1}\ \rangle \cong D_4,$$

where φ_1 is an amphicheiral map and φ_2 is an inversion map. Moreover, $[\varphi_1]^*$ is the orientation reversing isometry whose symmetric axis is orthogonal to that of ι as the following figure and $[\varphi_2]^*$ is the identity map.



3 Symmetry of twisted Alexander functions

We denote by $\Delta_{K,\rho}(t)$ the twisted Alexander invariant associated to α and $\rho \in \mathrm{Hom}(G_K, SU(2))$. (See for instance [W].)

On an arc component $A \subset \mathrm{Reg}(K)$ (resp. a circle component $C \subset \mathrm{Reg}(K)$), we fix a point $[\rho_0] \in A$ (resp. C). For any point $[\rho] \in A$ (resp. C), we choose a smooth path $\gamma_{[\rho]}$ from $[\rho_0]$ to $[\rho]$ and define a map $\phi_A\colon A \to \mathbb{R}$ (resp. $\phi_C\colon C \to \mathbb{R}/(\mathrm{Vol}\, C)\mathbb{Z}$) by

$$\phi_A([\rho]) := \int_{\gamma_{[\rho]}} \tau_K \quad \left(\text{resp. } \phi_C([\rho]) := \left[\int_{\gamma_{[\rho]}} \tau_K\right]\right).$$

Definition 3.1.

We write $\mathrm{Reg}(K) = (\coprod_{i=1}^{m} A_i) \amalg (\coprod_{j=1}^{n} C_j)$, where A_i is an arc component and C_j a circle component. Taking ϕ_{A_i} and ϕ_{C_j} as above, we set

$$\Delta_{A_i}(s,t) := \Delta_{K,\rho_s}(t)\ ,\ \ \inf \phi_{A_i} < s < \sup \phi_{A_i}$$
$$\Delta_{C_j}(s,t) := \Delta_{K,\rho_{[s]}}(t)\ ,\ \ [s] \in \mathbb{R}/(\mathrm{Vol}\, C)\mathbb{Z},$$

where $[\rho_s] = \phi_{A_i}^{-1}(s)$ and $[\rho_{[s]}] = \phi_{C_j}^{-1}([s])$.

For any component $D \subset \mathrm{Reg}(k)$, Δ_D is well-defined up to translation of the coordinate s and multiplication of t^{2n} for any $n \in \mathbb{Z}$.

Theorem 3.2.

(i)For any function $\Delta_D(s,t)$,

$$\Delta_{\iota(D)}(s,t) \doteq \Delta_D(-s,-t).$$

(ii)If there exists a class $[\varphi] \in \mathrm{Out}(G_K)$ such that $[\varphi]^*\colon \mathrm{Reg}(K) \to \mathrm{Reg}(K)$ is an orientation preserving (resp. reversing) isometry, then for any function $\Delta_D(s,t)$,

$$\Delta_{[\varphi]^*(D)}(s,t) \doteq \Delta_D(s,t) \quad (\textit{resp. } \Delta_{[\varphi]^*(D)}(s,t) \doteq \Delta_D(-s,t)).$$

"$\doteq$" means that the invariants equal to each other up to above ambiguity.

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