Third Homology Classes of Knot Quandle obtained from Shadow Coloured Diagrams

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1 Main results.

In this paper, we will construct a family of the third homology classes of link quandles, called the shadow diagram classes and the shadow fundamental classes. The constructions of them are motivated by the shadow cocycle invariants of links.

Let L be a link. As shown later, we can associate a third rack homology class $[D_x]$ with a pair of a diagram D of L and an element x of the link quandle Q_L, and a third quandle homology class $[L_x]$ with an element x. These two homology classes are called the shadow diagram class and the shadow fundamental class, respectively.

Our main results are the following ones:

Theorem 1.1. *When L is a non-trivial n-component link,*

a) the shadow diagram classes $[D_x]$ is a non-zero element of $H_3^{\mathrm{R}}(Q_L;\mathbb{Z})$;

b) there exist n distinct shadow fundamental class of L; and

c) the third quandle homology group $H_3^{\mathrm{Q}}(Q_L;\mathbb{Z})$ is splittable, that is,

$$H_3^{\mathrm{Q}}(Q_L;\mathbb{Z}) \cong \bigoplus \mathbb{Z}[L_i] \oplus (H_3^{\mathrm{Q}}(Q_L;\mathbb{Z})/\bigoplus \mathbb{Z}[L_i]),$$

where $[L_i]$ are n distinct shadow fundamental classes.

By using these shadow fundamental classes, we can characterise the shadow cocycle invariants in [CJKS] in a homological viewpoint.

Let X be a finite quandle and ϕ a 3-cocycle of X. We denote by $\Phi_\phi(L)$ the shadow cocycle invariant of a link L computed via ϕ. Considering the disjoint union $\tilde{L}$ of L and the unknot U, we have:

Corollary 1.2. *If X is a connected quandle,*

$$\Phi_\phi(L) = \sum_{f \in \mathrm{Hom}(Q_{\tilde{L}}, X)} \langle f_*[\tilde{L}_{\mathrm{sh}}], \phi \rangle$$

holds, where $\langle *, * \rangle \colon H_3^{\mathrm{Q}}(X) \otimes H^3_{\mathrm{Q}}(X) \to \mathbb{Z}$ is an evaluation map as usual, and $[\tilde{L}_{\mathrm{sh}}]$ is one of the shadow fundamental classes of $\tilde{L}$.

As for the based shadow cocycle invariants defined in [S], we have:

Corollary 1.3. *If X is a connected quandle,*

$$\Phi^*_\phi(L) = \sum_{f \in \mathrm{Hom}(Q_L, X)} \langle f_*[L_{\mathrm{sh}}], \phi \rangle$$

holds, where $[L_{\mathrm{sh}}]$ is one of the shadow fundamental classes of L.

Directly from these two corollaries, we obtain:

Corollary 1.4. *For a connected quandle X,*

$$\Phi_\phi(L) = |X| \cdot \Phi^*_\phi(L)$$

holds.

We notice that this corollary is a generalisation of Satoh's result in [S].

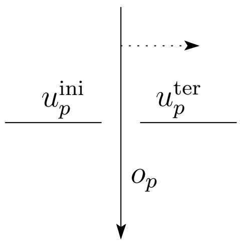
2 Sketch of Proof of Theorem 1.1.

First we recall the Wirtinger presentation of link quandle Q_L of a link L.

Let D be a link diagram of L. The set of arcs of D is denoted by $\mathcal{A}(D)$ and the set of crossings by $\mathcal{C}(D)$. The Wirtinger presentation of Q_L consists of $\mathcal{A}(D)$ as the set of generators and the set of relations

$$u_p^{\mathrm{ini}} \lhd o_p = u_p^{\mathrm{ter}}$$

for each $p \in \mathcal{C}(D)$, where o_p, u_p^{ini} and u_p^{ter} are arcs at p as depicted in Figure 2.1. Thus we can consider the diagram D as cannonically coloured by Q_L.



$$u_p^{\mathrm{ini}} \lhd o_p = u_p^{\mathrm{ter}}$$

Figure 2.1: Notation of arcs at a crossing p.

Since the quandle coloured diagram on $\mathbb{S}^2$ can be shadow colourable, we can also consider D as a Q_L shadow coloured diagram. From this fact, we obtain a third rack homology class $[D_x]$ of Q_L, where x is a shadow colour of the region containing basepoint $*$ which is supposed to be sufficiently distant from L. This is the shadow diagram class of D, and the image $[L_x] = \rho_*[D_x]$ is called the shadow fundamental class of L, where $\rho_*\colon H_3^{\mathrm{R}}(Q_L) \to H_3^{\mathrm{Q}}(Q_L)$ is a cannonical homomorphism.

Next, we will prove the invariance of $[L_x]$ under the choice of diagrams D. This is easily calculated. We also prove that $[L_x] = [L_y]$ when x and y are in the same orbit of Q_L. Our proof is to construct a 4-chain diagrammatically whose boundary is equal to $\langle D_x - D_y \rangle$. Figure 2.2 shows a Q_L-shadow coloured 3-diagram representing the 4-chain. This diagram consists of $D \times I$ and n copies $S_1, \ldots, S_n$ of $\mathbb{S}^2$. By giving shadow colours to the regions of this diagram, we can see that $[L_x]$ and $[L_y]$ coincide.

Finally, we prove that, if x and y do not belong to the same orbit, $[L_x]$ and $[L_y]$ are distinct. If these two homology classes are coincident, there exists a 4-chain whose boundary is equal to $\langle D_x - D_y \rangle$. Also there exists a Q_L shadow coloured 3-diagram representing this 4-chain. Thus we conclude that x and y belong to the same orbit, which contradicts the assumption.

Now we obtain Theorem 1.1.

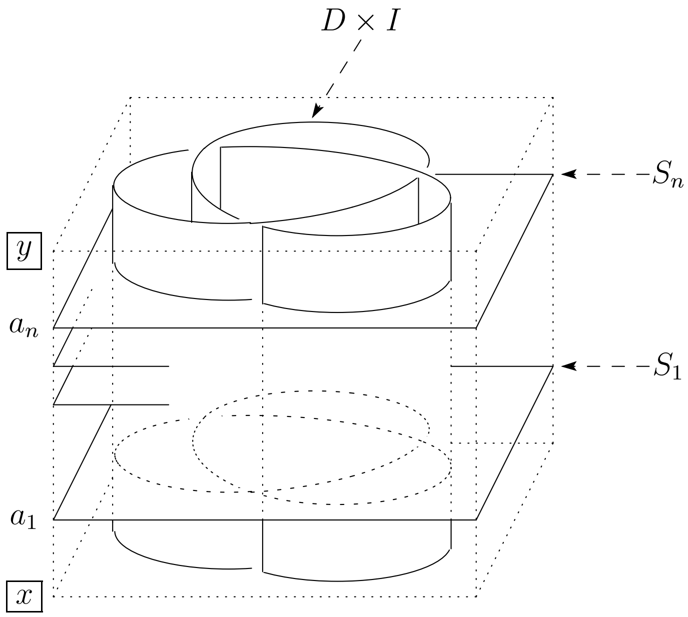


Figure 2.2: Construction of 3-chain.

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