# A $G$-family of quandles and cocycle invariants for handlebody-links

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## • Handlebody-link

A *handlebody-link* $L$ is a disjoint union of circles and a finite trivalent graph embedded in the 3-sphere $S^3$. An *IH-move* is a local change of a handlebody-link as described in Figure 1, where the replacement is applied in a disk embedded in $S^3$. Two handlebody-links are *equivalent* if they are related by a finite sequence of IH-moves and isotopies of $S^3$. The equivalence class is equivalent to the isotopy class of handlebodies embedded in $S^3$.

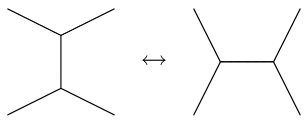

Figure 1:

**Theorem 1** (Ishii)**.** *Two handlebody-link diagrams represent an equivalent handlebody-link if and only if they are related by a finite sequence of the moves R1–6 depicted in Figure 2.*

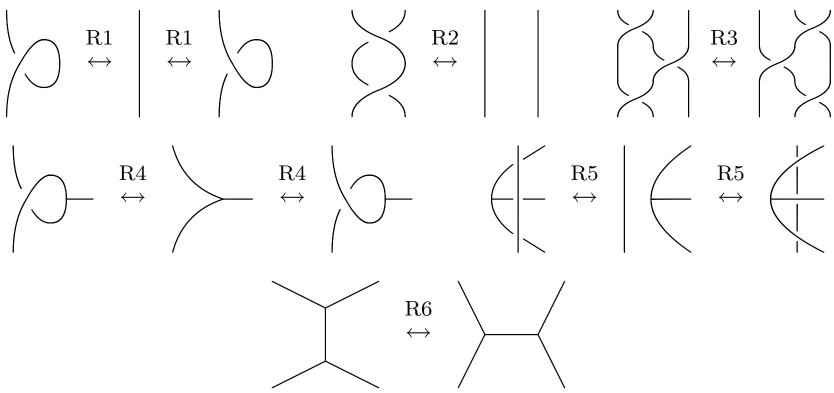


Figure 2:

We define a coloring of a handlebody-link diagram by

| | | |
|---|---|---|
| $(X, *, \iota)$-set (colors of regions) | symmetric quandle $(X, *, \iota)$ (colors of arcs) | $G$-family of quandles (colors around vertices) |

## • Quandle, symmetric quandle and $G$-family of quandles

A *quandle* $(X, * : X \times X \longrightarrow X)$ is a non-empty set $X$ with a binary operation $* : X \times X \longrightarrow X$ satisfying the following axioms:

- For any $a \in X$, $a * a = a$.
- A map $*a : X \longrightarrow X$ defined by $*a(b) = b * a$ is a bijection for any $a \in X$.
- For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

An $(X,*,\iota)$ is a *symmetric quandle* if $(X,*)$ is a quandle and $\iota$ is an involution of $X$ such that

$$\iota(a*b)=\iota(a)*b, \quad a=(a*(\iota(b)))*b, \qquad \text{for any } a,b\in X.$$

**Example 2.**
(1) When $X=\mathbf{Z_n}(\cong \mathbf{Z}/n\mathbf{Z}), a*b=2b-a, \iota(a)=a$, $(X,*)$ is called a *dihedral quandle* and $(X,*,\iota)$ is a symmetric quandle.
(2) When $X$ is a group $G$, $g*h=h^{-1}gh, \iota(g)=g^{-1}$, $(G,*)$ is called a *conjugation quandle* and $(G,*,\iota)$ is a symmetric quandle.

A set $(Y,\tilde{*}:Y\times X\longrightarrow Y)$ is an $(X,*,\iota)$*-set* if it is satisfied the following conditions:

- A map $\tilde{*}a:Y\longrightarrow Y$ defined by $\tilde{*}a(b)=b\tilde{*}a$ is a bijection for any $a\in X$.
- For any $a\in Y, b,c\in X, (a\tilde{*}b)\tilde{*}c=(a\tilde{*}c)\tilde{*}(b*c)$.
- For any $a\in Y, b\in X, (a\tilde{*}\iota(b))\tilde{*}b=a$.

**Example 3.**
(1) When $Y=X, a\tilde{*}b:=a*b$, $(X,\tilde{*})$ is an $(X,*,\iota)$-set.
(2) When $Y=\{y\}, y\tilde{*}a:=y$, $(Y,\tilde{*})$ is an $(X,*,\iota)$-set.

Let $G$ be a finite quandle. A set $(X,*^G:X\times X\times G\longrightarrow X)$ is a *G-family of quandles* if it is satisfied the following conditions;

- $a*^g a=a$,
- $a*^e b=a$,
- $(a*^g b)*^h b=a*^{gh} b$,
- $(a*^g b)*^h c=(a*^h c)*^{h^{-1}gh}(b*^h c)$,

for any $a,b,c\in X, g,h\in G$ where $e$ is the identity of $G$. Here, $a*^g b$ means $*^G(a,b,g)$. Since a set $(X,*^g)$ is a quandle for each $g\in G$, $(X,*^G)$ gives a family of quandles.

**Example 4.**
Let $G$ be a finite group generated by one element, i.e., $G=\mathbf{Z}_m$ for some $m$. $X$ is a quandle of type $m$, i.e., $X$ is a quandle such that $m=\min\{k>0|\underbrace{(\cdots(a*b)\cdots)*b}_{k's}=a \text{ for any } a,b\in X\}$. Then, $(X,*^{\mathbf{Z}_m})$ is a $\mathbf{Z}_m$-family of quandles where $a*^1 b:=a*b, a*^i b:=(a*^{i-1}b)*b$.

A set $(Y,\tilde{*}^G:Y\times X\times G\longrightarrow Y)$ is an $(X,*^G)$*-set* it it is satisfied the following conditions;

- $y\tilde{*}^e a=y$,
- $(y\tilde{*}^g a)\tilde{*}^h a=y\tilde{*}^{gh}a$,
- $(y\tilde{*}^g a)\tilde{*}^h b=(y\tilde{*}^h b)\tilde{*}^{h^{-1}gh}(a*^h b)$,

for any $y\in Y, a,b\in X, g,h\in G$ where $e$ is the identity of $G$.

**Example 5.**
(1) When $Y=X, a\tilde{*}^g b:=a*^g b$, $(X,\tilde{*}^G)$ is an $(X,*^G)$-set.
(2) When $Y=\{y\}, y\tilde{*}^g a:=y$, $(Y,\tilde{*}^G)$ is an $(X,*^G)$-set.

**Proposition 6.** *Let $(X,*^G)$ be a $G$-family of quandles. Let $*$ be a binary operation of $X\times G$ defined by $(a,g)*(b,h):=(a*^h b, h^{-1}gh)$ and $\iota$ be an involution of $X\times G$ defined by $\iota(a,g):=(a,g^{-1})$. Then, $(X\times G,*,\iota)$ is a symmetric quandle. Futhermore, let $(Y,\tilde{*}^G)$ be an $(X,*^G)$-set. Then, $(Y,\tilde{*})$ is an $(X\times G,*,\iota)$-set where $y\tilde{*}(a,g):=y\tilde{*}^g a$.*

# • Coloring of a handlebody-link diagram

Let $(X, *^G)$ be a $G$-family of quandles and $(Y, \tilde{*}^G)$ be an $(X, *^G)$-set. We define a coloring of a handlebody-link diagram via a symmetric quandle $(X \times G, *, \iota)$ and an $(X \times G, *, \iota)$-set defined by $(X, *^G)$ and $(Y, \tilde{*}^G)$ as in Proposition 6.

Let $D$ be a diagram of a handlebody-link $L$. We assign an element of $(X \times G) \times \{\text{arc orientation}\}/(\alpha, o) \cong (\iota(\alpha), -o)$ to each arc and an element of $Y$ to each region satisfying the conditions as in Figure 3. Then, such an assignment is called a $(Y, \tilde{*})$-*shadow* $(X \times G, *, \iota)$-*coloring* of $D$. A quandle coloring is well-defined and we have the following theorem. We denote the set of $(Y, \tilde{*})$-shadow $(X \times G, *, \iota)$-colorings of $D$ by $Col_{X \times G, Y}(D)$.

**Theorem 7.** *Let $D, D'$ be handlebody-link diagrams such that $D'$ is obtained by applying one of the move R1-R6 to $D$ once. Then, for $C \in Col_{X \times G, Y}(D)$, there is a unique $C_{D,D'} \in Col_{X,Y}(D')$ such that colors are preserved outside of the neiborhood in which the move is applied.*

This theorem implies that the number of elements in $Col_{X \times G, Y}(D)$ is an invariant of a handlebody-link $L$.

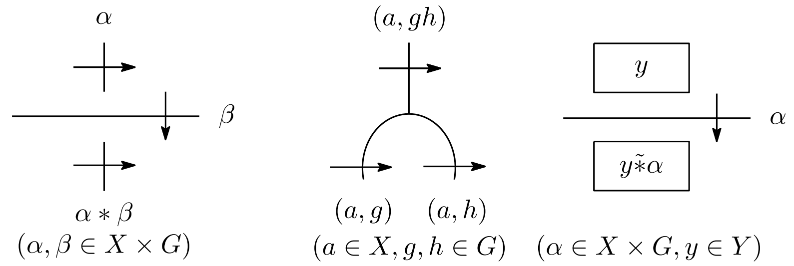


Figure 3:

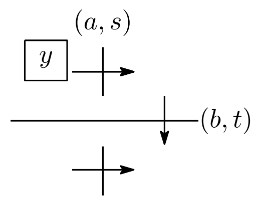


Figure 4:

# • Cocycle invariants

The quandle homology theory and cohomology theory were introduced by Carter, Jelsovsky, Kamada, Langford and Saito to define invariants of oriented classical links and knotted surfaces. We denote the homology gruop and the cohomology group by $H_n^Q(X; A)$ and $H_Q^n(X; A)$ where $A$ is an abelian gruop. The authors introduced new homology thoery and cohomology theory to define invariants of handlebody-links associated with colorings defined above section when a $G$-family of quandles is $(X, *^G)$ given in Example 4. We denote the homology gruop and the cohomology group by $H_n(X; A)_Y$ and $H^n(X; A)_Y$ where $A$ is an abelian gruop.

Set $[\theta] \in H^2(X; A)_Y$. Let $D$ be a diagram of a handlebody-link $L$. For $C \in Col_{X \times \mathbf{Z}_m, Y}(D)$, the Boltzmann weight $B(\tau, C) \in A$ at each crossing $\tau$,

$$B(\tau; C) = \sum_{i=0}^{s-1} \sum_{j=0}^{t-1} \theta((y \tilde{*}^i a) \tilde{*}^j b, a *^j b, b).$$

Here, $y, (a, s), (b, t)$ are colors of the region and the arcs around $\tau$ as in Figure 4.

**Theorem 8.** *Let $D$ be a diagram of a handlebody-link $L$. Then, the multi-set $\{\sum_{\tau: crossing} B(\tau; C) | C \in Col_{X \times \mathbf{Z}_m, Y}(D)\}$ is an invariant of $L$. Furthermore, this invariant is independent of a choice of a represent element of $[\theta]$.*

The following theorem is useful to find cocycles.

**Theorem 9.**

1. *If $[\theta] \in H_Q^2(X; A)$ satisfies that $\sum_{i=0}^{m-1} \theta(a *^i b, b) = m\theta(a, b) = 0$, then we have $[1 \otimes \theta] \in H_Q^2(X; A)_{\{y\}}$.*

2. *If $[\theta] \in H_Q^3(X; A)$ satisfies that $\sum_{i=0}^{m-1} \theta(a *^i b, b, c) = \sum_{i=0}^{m-1} \theta(a *^i c, b *^i c, c) = 0$, then we have $[\theta] \in H_Q^2(X; A)_X$.*