Braid ordering, Nielsen-Thurston classification and geometry of knot complements

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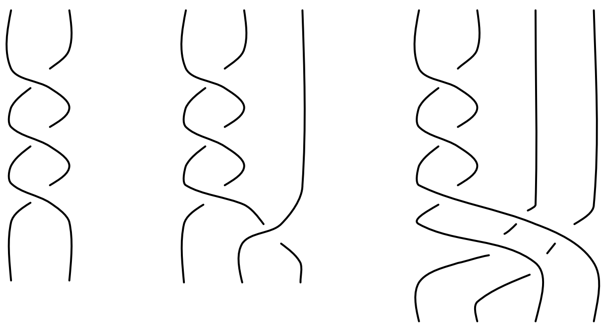
1 Introduction

B_n: Braid group, $\widehat{\beta}$: Closure of braid β.
$\Delta = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\sigma_2\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)(\sigma_1)$: Garside's fundamental braid.

- Nielsen-Thurston theorem states braids $\beta \in B_n$ are classified into following three types by their dynamics:
  1. *periodic* : some powers of β is equal to Δ^m for some integer m.
  2. *reducible*: its representing homeomorphism is reducible.
  3. *pseudo-Anosov* : its representing homeomorphism is pseudo-Anosov.
- On the other hand, Thurston also shows knots are classified into following three types by the geometry of its complements.
  1. *Torus knot* : knots which can be positioned to canonically embedded torus.
  2. *Satellite knot* : its complement contains essential torus.
  3. *Hyperbolic knot*: its complement admits complete hyperbolic structure with finite volume.

Unfortunately, these classifications are not in one-to-one correspondence.



- **When the Nielsen-Thurston classification determines the geometry of the complements of closed braids?**

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2 Braid ordering, Dehornoy floor

Definition 1. (Dehornoy ordering)

For $\alpha, \beta \in B_n$, we say that $\alpha < \beta$ is true if $\alpha^{-1}\beta$ admits a braid word representative which contains σ_i, and does not contain $\sigma_1^{\pm 1}, \sigma_2^{\pm 1} \cdots, \sigma_{i-1}^{\pm 1}, \sigma_i^{-1}$ for some i. The relation $<$ defines left-invariant total ordering on B_n. We call this ordering on B_n *Dehornoy ordering.*

Definition 2. (Dehornoy floor)

The *Dehornoy floor* $[\beta]_D$ of braid $\beta \in B_n$ is a positive integer such that:

$$\Delta^{2[\beta]_D} \leq \beta < \Delta^{2[\beta]_D+2} \text{ if } \beta > 1, \Delta^{-2[\beta]_D-2} < \beta \leq \Delta^{-2[\beta]_D} \text{ if } \beta \leq 1$$

- The Dehornoy floor is a measure of complexity in terms of braid ordering.
- The Dehornoy floor is efficiently computable for each braid.

Proposition 1 (Properties of Dehornoy floor ([MN]))**.** *Let β be a braid. Then following holds.*

1. *If a braid $\beta \in B_n$ is represented by a braid word which contains s occurrence of σ_1 and k occurrence of σ_1^{-1}, then $[\beta]_D < max\{s, k\}$.*
2. *$|[\beta]_D - [\beta']_D| \leq 1$ if β and β' are conjugate.*
3. *For every n, there exists a positive integer $r(n)$ such that for every braid $\beta \in B_n$, if $[\beta]_D > r(n)$ then $\widehat{\beta}$ is the unique closed representative of its closure in B_n. That means, if $\widehat{\beta} = \widehat{\beta}'$ is true for some $\beta' \in B_n$, then β and β' are conjugate.*

3 Main results

Theorem 1 (Main Theorem)**.** *Let $\beta \in B_n$ be a braid whose closure is a knot. If $[\beta]_D \geq 3$, following holds:*

1. *β is periodic if and only if $\widehat{\beta}$ is a torus knot.*
2. *β is reducible if and only if $\widehat{\beta}$ is a satellite knot.*
3. *β is pseudo-Anosov if and only if $\widehat{\beta}$ is a hyperbolic knot.*

- When the number of strand are prime, the situation becomes much simpler.

Corollary 1. *Let p be a prime and $\beta \in B_p$ be a braid whose closure is a knot and $[\beta]_D \geq 3$. Then $\widehat{\beta}$ is hyperbolic if and only if β is non-periodic.*

As a corollary, we obtain almost disjoint infinite family of hyperbolic knots for each pseudo-Anosov elements of mapping class group of punctured disc. Let $\pi : B_n \to MCG(D_n)$ be a natural projection between braid group and the mapping class group of n-punctured disc.

Corollary 2. *Suppose* $[f] \in MCG(D_n)$ *be a pseudo-Anosov element and let* $P([f]) = \{\widehat{\beta} | \beta \in \pi^{-1}([f]), [\beta]_D \geq 3\}$.

Then $P([f])$ *consists of infinite number of distinct hyperbolic knots. Moreover for another pseudo-Anosov element* $[g] \in MCG(D_n)$*, if* $[g]$ *is not conjugate to* $[f]$*, then the intersection of* $P([f])$ *and* $P([g])$ *are finite.*

4 Idea of proof

- Torus knots and reducible braids are well-understood ([Me]), we can easily prove equivalence of them.
- Closure of reducible braids are satellite knots. To prove the converse, we investigate essential torus T.
- Birman-Menasco's braid foliation theory ([BM]) and our assumption shows T can be positioned to "standard position" .
- By seeing how T interferes braidings of braid strands, we see only possible position of T is type 0: That is, the core of T is a closed braid, hence satellite knot is represented by reducible braid.
- Both classifications are exclusive, we obtain equivalence of hyperbolic knots and pseudo-Anosov braids.

References

[BM] J.Birman, W.Menasco, *Special positions for essential tori in link complements,* Topology, **33**, No.3 (1994), 525-556.

[DDRW] P. Dehornoy, I. Dynnkov, D. Rolfsen and B. Wiest, *WHY ARE THE BRAIDS ORDERABLE ?,* Panoramas et Synthéses **14**, Soc. Math. France. 2002.

[I1] T.Ito, *Application of braid ordering to knot theory I: Essential surface in link complement,* In preparation.

[I2] T.Ito, *Application of braid ordering to knot theory II: Braid dynamics and the geometry of closed braid complements,* In preparation.

[Me] W.Menasco, *On iterated torus knots and transversal knots,* Geometry & Topology, **5**, (2001), 651-682.

[MN] A.Malyutin, N.Netsvetaev, *Dehornoy's ordering on the braid group and braid moves,* St.Peterburg Math. J. **15**, No.3 (2004), 437-448.