トポロジー火曜セミナー

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Introduction

Khovanov homology

[Khovanov, 2000] A bigraded **chain complex** $C^{*,\star}(D)$ (of abelian groups) for each link diagram D so that

$$Kh^{i,j}(D) := H^i(C^{*,j}(D))$$

is **invariant under Reidemeister moves**. This is nowadays called **Khovanov homology**.

$\rightsquigarrow$ Relation to **TQFT** was pointed out.

Theorem 1.1 ([Kronheimer and Mrowka, 2011])
Khovanov homology **detects the unknot**.

[Lee, 2005] constructed a **variant**.

$\rightsquigarrow$ a **concordance invariant** [Rasmussen, 2010].

[Bar-Natan, 2005] Khovanov complex in terms of **cobordisms**.

$\rightsquigarrow$ Invariants for **tangles** instead of links.

$\rightsquigarrow$ Changing **TQFT**, we get **variants** including **Lee homology** and **Bar-Natan homology**.

$\rightsquigarrow$ **"universal" Khovanov homology**.

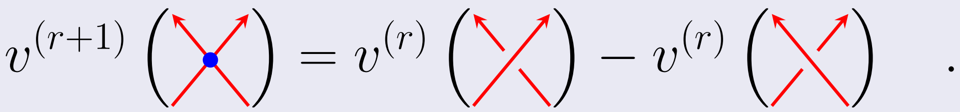
Vassiliev derivative

$\mathcal{X}^{(r)}$: the set of **singular links** with **exactly r double points**.

Definition ([Vassiliev, 1990] (implicit), [Birman, 1993], [Birman and Lin, 1993])

v: a knot invariant with values in A.

$\rightsquigarrow v^{(r)} : \mathcal{X}^{(r)} \to A$ by $v^{(0)} := v$ and **Vassiliev skein relation**:



We call $v^{(r)}$ the **r-th (Vassiliev) derivative** of v.

Definition

v is called **of finite type** (or **Vassiliev type**) if $v^{(r)} \equiv 0$ for $r \gg 0$.

Slogan Finite-type invariants are **polynomials**.

cf. [Volić, 2006, Budney et al., 2017].

Theorem 1.2 ([Birman, 1993, Birman and Lin, 1993])

The **Taylor coefficients** *of the Jones polynomial* **at $t = 1$** *are of finite type.*

Question

Any relations between Khovanov homology and **finite type invariants**?

First goal

To understand Khovanov homology in view of **Vassiliev theory**.

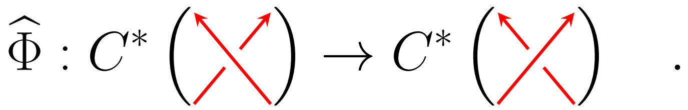
Vassiliev derivatives of knot homologies

Question

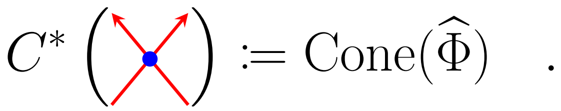
What are Vassiliev derivatives of knot homologies?

Strategy

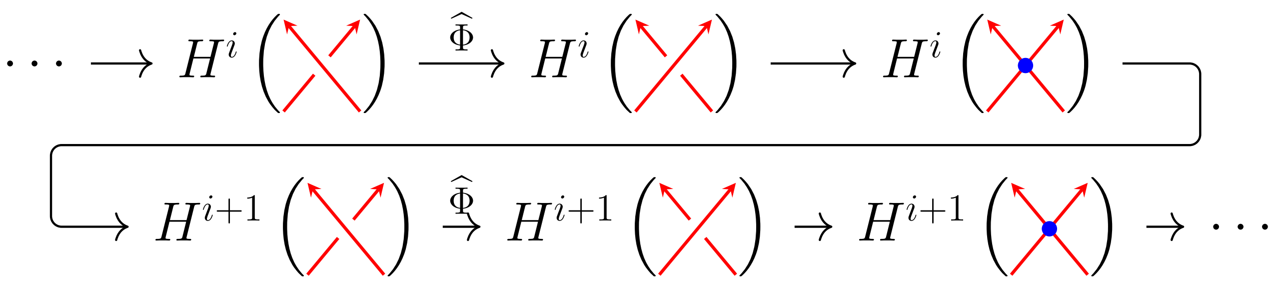
1. Realize a crossing change as a morphism of chain complexes:



⇝ For singular diagrams, take **mapping cones** recursively:



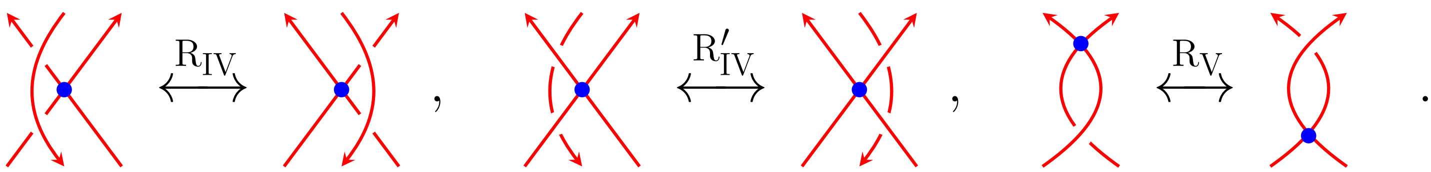
⇝ A **categorified Vassiliev skein relation**:



Remark

The long exact sequence above yields the **ordinary Vassiliev skein relation** on the Euler characteristics.

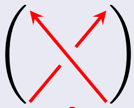
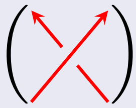
2. Check invariance under **moves of double points**:

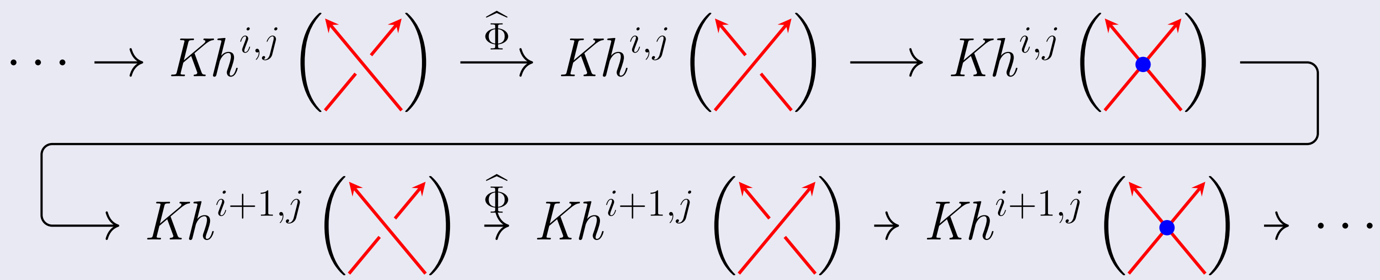


Main result I

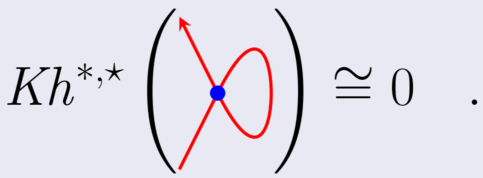
Main Theorem I (Ito, Y.)

Khovanov homology $Kh(-)$ *extends to a* **singular link invariant** *so that*

1. *there is a morphism* $\widehat{\Phi} : Kh^{i,j}$  $\to Kh^{i,j}$  *together with a* **categorified Vassiliev skein relation**



2. *the following categorified version of* **FI relation***:*



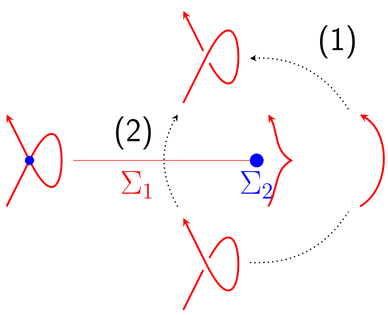
Remark

The morphism $\widehat{\Phi}$, called the **genus-one morphism**, is different from the **concordance theoretic crossing-change** (e.g. see [Hedden and Watson, 2018]).

In fact, $\widehat{\Phi}$ is the **first concrete instance** of non-trivial maps of bidegree $(0, 0)$.

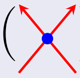
Meaning of FI relation

FI relation arises from comparison of the following two "paths:"



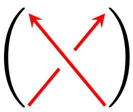
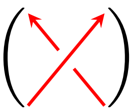
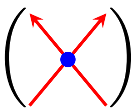
Main result II

Question

Can Vassiliev derivatives $Kh($$)$ be computed independently of the resolutions?

Motivation

- $\mathrm{Cone}(\widehat{\Phi})$ is large.
  $\rightsquigarrow$ Difficult to compute examples.
- If two of the three homologies

  Kh  , Kh  , and Kh 

  are computed, the other may also be determined thanks to the **categorified Vassiliev skein relation**.

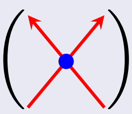
Main Theorem II (Y. arXiv:2007.15867)

D: a **singular link diagram** *with* **exactly one double point**.

$\rightsquigarrow$ *There is a chain complex $C^{*,\star}_{\mathsf{crx}}(D)$, called the* **crux complex**, *together with a (graded) endomorphism*

$$\Xi : C^{*-2,\star-2}_{\mathsf{crx}}(D) \to C^{*+2,\star+4}_{\mathsf{crx}}(D)$$

such that

C_{Kh}  $\simeq \mathrm{Cone}(\Xi)$.

Slogan Main Theorem II computes the 1^{st} **Vassiliev derivative** of $Kh(-)$.

Today's plan

Khovanov homology

The category $\mathcal{C}ob_2^{\ell}(Y_0, Y_1)$

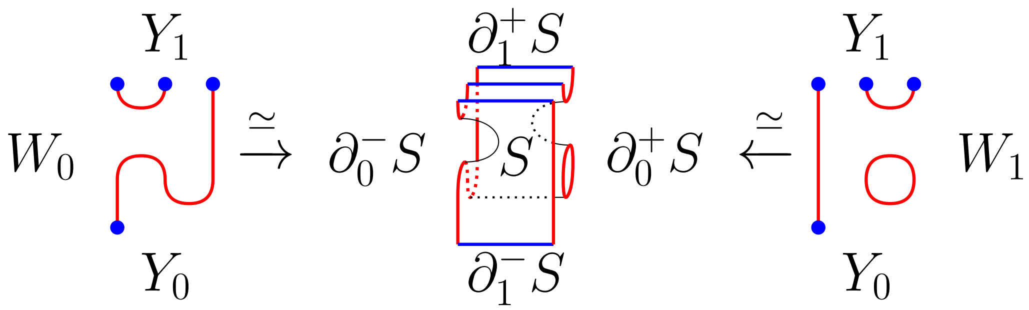
Y_0, Y_1: compact oriented 0-manifolds.

Definition

Define $\mathbf{Cob}_2(Y_0, Y_1)$ to be a category such that

- objects are (oriented) **1-cobordisms** $W : Y_0 \to Y_1$;
- morphisms are (diffeo. classes of) **2-cobordisms with corners** (aka. **2-bordisms**).
- composition is given in terms of **gluing**.

2-bordisms $S : W_0 \to W_1 \in \mathbf{Cob}_2(Y_0, Y_1)$:



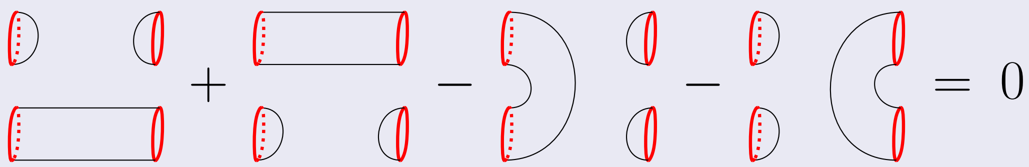
Definition

Define $\mathcal{C}ob_2^{\ell}(Y_0, Y_1)$ to be the **k-linear additive category** generated by $\mathbf{Cob}_2(Y_0, Y_1)$ subject to the following relations:

S-relation $S \amalg S^2 \sim 0$ for $S : W_0 \to W_1$;

T-relation $S \amalg T^2 \sim 2 \cdot S$ for $S : W_0 \to W_1$;

$4Tu$-relation



Remark

The morphisms of $\mathcal{C}ob_2^{\ell}(Y_0, Y_1)$ are graded by **Euler characteristics**.

$\mathcal{C}ob_2^{\ell}(Y_0, Y_1)$: a **graded k-linear category**.

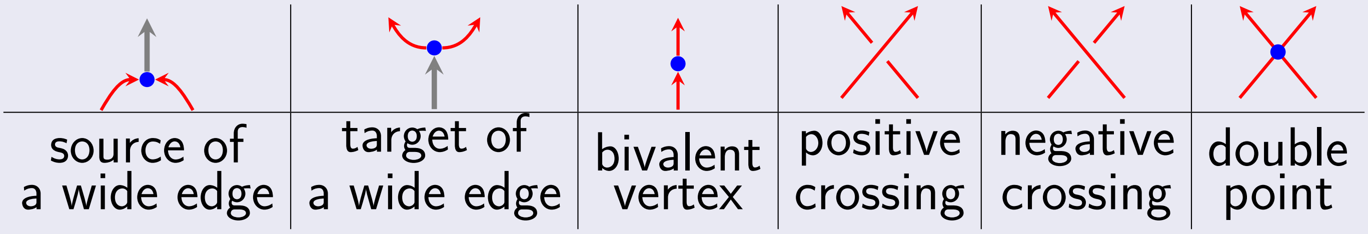
Singular link-like graph

Definition

A **singular link-like graph** is a planar graph $G = \{E(G) \rightrightarrows V(G)\}$, say $V^r(G)$ the set of r-valent vertices, together with data

- a subset $c(G) \subset V^4(G)$ of **crossings** with **signs**;
  $\rightsquigarrow$ elements of $c^\sharp(G) := V(G) \setminus c(G)$: **double points**;
- a subset $E^{\mathsf{w}}(G) \subset E(G)$ of **wide edges**;

such that each vertex is locally depicted as follows:



In particular, a **singular link diagram** is nothing but a singular link-like graph without wide edges.

Convention

- Vertices of the left three types are omitted from pictures.
- Bivalent vertices are removed whenever possible.

Remark

If $V^4(G) = \varnothing$, then the **union of non-wide edges** is a **smooth 1-manifold**.

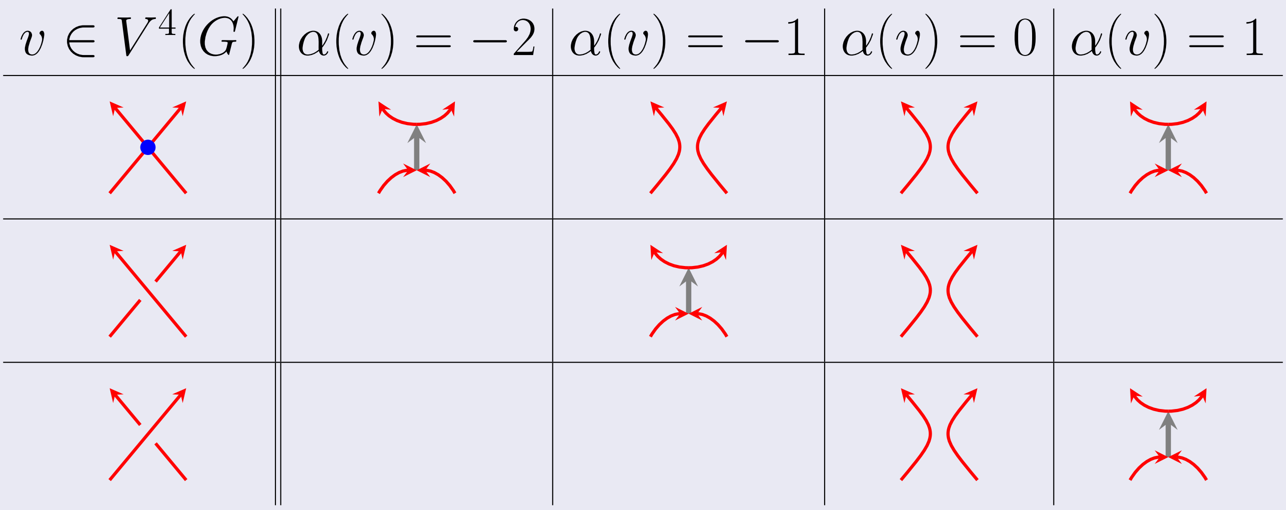
Smoothings of link-like graphs

Definition

A map $\alpha : V^4(G) \to \mathbb{Z}$ is said to **lie in the effective range** if

- $0 \le \alpha(v) \le 1$ for positive crossings v;
- $-1 \le \alpha(v) \le 0$ for negative crossings v;
- $-2 \le \alpha(v) \le 1$ for double points v.

In this case, define the **α-smoothing** G_α by replacing quadri-valent vertices as follows:



Definition

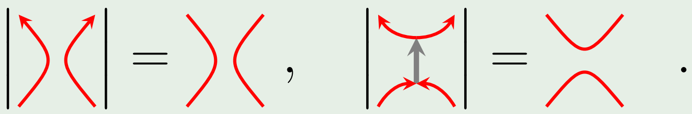
For each $\alpha : V^4(G) \to \mathbb{Z}$, define $|G_\alpha| \in \mathcal{C}ob^{\ell}_2(\varnothing, \varnothing)$ as follows:

α:eff. $|G_\alpha|$: the union of non-wide edges of G_α

$\rightsquigarrow |G_\alpha|$ (+ orientation) is an object of $\mathbf{Cob}_2(\varnothing, \varnothing)$.

α:non-eff. $|G_\alpha| = 0$.

Example



Remark

Precisely, we need a **checkerboard coloring** for G to determine the orientation on $|G_\alpha|$. Details are omitted in this talk.

Multi-fold complexes

Idea: construct **Khovanov complex** of G by **categorically** summing up all **states** (i.e. α in the effective range).

Khovanov discussed **cubes** of states.

$\rightsquigarrow$ We generalize them to consider **double points**.

$\mathcal{A}$: an additive category, S: a finite set.

Definition

An **S-fold complex** in $\mathcal{A}$ consists of

- a family of objects $\{X^\alpha\}_\alpha$ of $\mathcal{A}$ indexed by elements α of the free abelian group generated by S;
- for each element $a \in S$, a morphism $d_a = d_a^\alpha : X^\alpha \to X^{\alpha+1}$;

which satisfy the following relations:

$$d_a^2 = 0 , \quad d_a d_b = d_b d_a \quad (a \neq b) \quad .$$

Definition

$X^\bullet$, $Y^\bullet$: S-fold complexes in $\mathcal{A}$.

$\rightsquigarrow$ a **morphism** $f : X^\bullet \to Y^\bullet$ **of S-fold complexes** consists of a morphism $f^\alpha : X^\alpha \to Y^\alpha$ for each $\alpha \in \mathbb{Z}S$ with $f d_a = d_a f$.

$\rightsquigarrow$ $\mathbf{MCh}_S(\mathcal{A})$: the category of S-fold complexes.

Remark

If $\mathcal{A}$ is additive (resp. abelian), then so is $\mathbf{MCh}_S(\mathcal{A})$.

Example

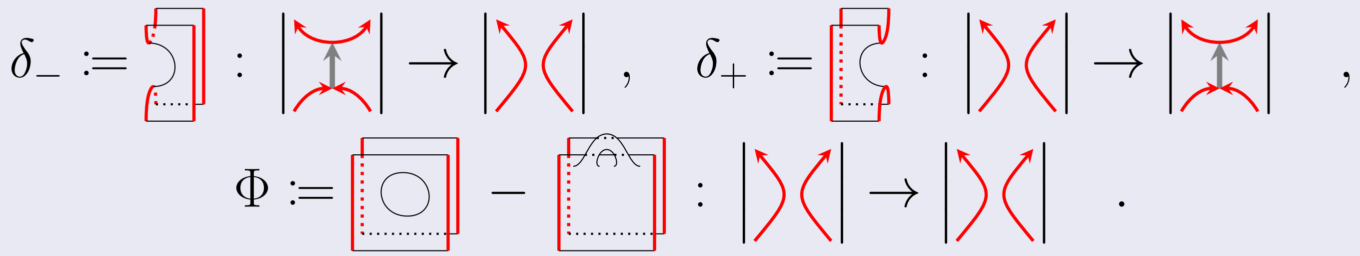
1. For $S = \{*\}$, $\mathbf{MCh}_{\{*\}}(\mathcal{A}) \cong \mathbf{Ch}(\mathcal{A})$.
2. If $S = \{\mathsf{H}, \mathsf{V}\}$, $\mathbf{MCh}_{\{\mathsf{H},\mathsf{V}\}}(\mathcal{A})$ is identified with the category of **bicomplexes** in $\mathcal{A}$.
3. If $S = S_1 \amalg S_2$, then there is a canonical equivalence

$$\mathbf{MCh}_{S_1 \amalg S_2}(\mathcal{A}) \simeq \mathbf{MCh}_{S_1}(\mathbf{MCh}_{S_2}(\mathcal{A})) \quad .$$

The multi-fold complex of smoothings

Definition

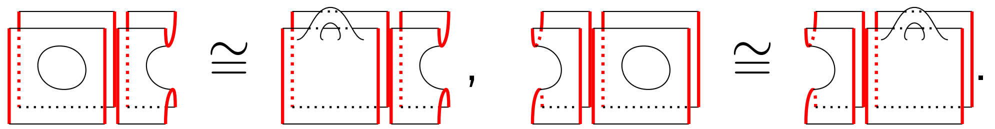
Define three morphisms in $\mathcal{Cob}_2^{\ell}(-,-)$ as follows:



Lemma 2.1

$\Phi\delta_- = 0$ *and* $\delta_+\Phi = 0$

Proof

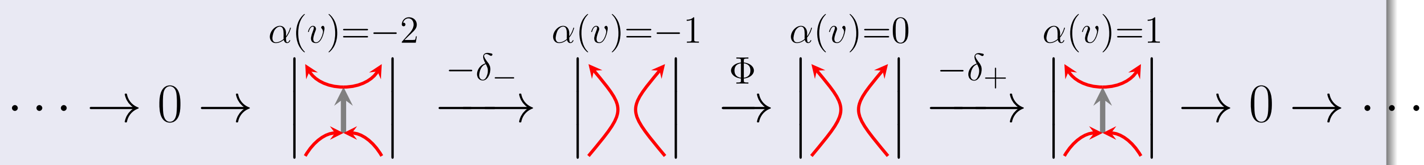


□

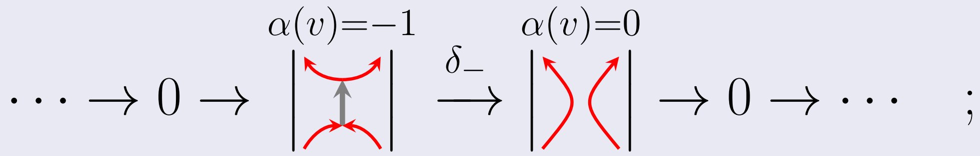
Definition

Define a $V^4(G)$-fold complex $\mathrm{Sm}(G)^\bullet$ in $\mathcal{Cob}_2^{\ell}(\varnothing,\varnothing)$ by $\mathrm{Sm}(G)^\alpha := |G_\alpha|$ with differentials given as follows:

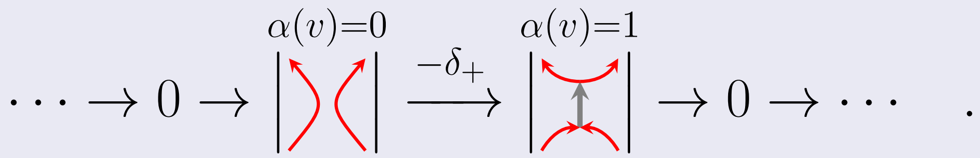
- if v is a double point, d_v is given by



- if v is a negative crossing, d_v is given by



- if v is a positive crossing, d_v is given by



Universal Khovanov complexes

Total complexes of bicomplexes:

$$\begin{array}{ccccc} \{\text{bicomplexes}\} & \xrightarrow{\text{anti-comm.}} & \{\text{double complexes}\} & \xrightarrow{\mathrm{Tot}} & \{\text{complexes}\} \\ X^{i,j} & \longmapsto & X^{i,j} & \longmapsto & \bigoplus_{i+j=n} X^{i,j} \\ d_{\mathsf{H}},\ d_{\mathsf{V}} & \longmapsto & d_{\mathsf{H}},\ (-1)^i d_{\mathsf{V}} & \longmapsto & \sum d_{\mathsf{H}} + (-1)^i d_{\mathsf{V}} \end{array}$$

Definition

S: a **totally ordered set**.

$\rightsquigarrow$ For a **bounded** S-fold complex X, define $\mathrm{Tot}(X)$ as a complex given by

$$\mathrm{Tot}(X)^n := \bigoplus_{|\alpha|=n} X^\alpha \ , \quad d_{\mathrm{tot}} := \sum_{a\in S} (-1)^{\sum_{b<a}\alpha(b)} d_a \quad ,$$

here $|\alpha| := \sum_a \alpha(a)$.

Remark

The isomorphism type of $\mathrm{Tot}(X)$ does not depend on total orders on S. In fact, there is a "universal" sign convention.

Definition

For a singular link-like graph G, we define the **universal Khovanov complex** as

$$[\![G]\!] := \mathrm{Tot}(\mathrm{Sm}(G)^\bullet) \quad .$$

Fundamental cofiber sequences

Definition

$X^\bullet$: an S-fold complex.

- For $\alpha_0 \in \mathbb{Z}S$, we write $X[\alpha_0]^\bullet$ the S-fold complex with

$$X[\alpha_0]^\alpha := X^{\alpha-\alpha_0} , \quad d_{X[\alpha_0],a} := (-1)^{\alpha_0(a)} d_{X,a} \quad .$$

  $X[\alpha_0]$ is called the **shift** of $X^\bullet$ by α_0.
- For $a \in S$ and $r \in \mathbb{Z}$, we write $\sigma_a^{\leq r} X^\bullet$ (resp. $\sigma_a^{\geq r}$, etc...) the S-fold complex given by

$$\sigma_a^{\leq r} X^\alpha := \begin{cases} X^\alpha & \alpha(a) \leq r \,(\text{resp. } \alpha(a) \geq r, \text{etc...}), \\ 0 & \text{otherwise}, \end{cases}$$

$$d^\alpha_{\sigma_a^{\leq r} X,a} := \begin{cases} d^\alpha_{X,a} & \alpha(a) \leq r-1 \,(\text{resp. } \alpha(a) \geq r, \text{etc...}), \\ 0 & \text{otherwise.} \end{cases}$$

  $\sigma_a^{\leq r} X^\bullet$ is called the **stupid truncation** of $X^\bullet$ along a at r.

Proposition 2.2

$X^\bullet$: a bounded S-fold complex, $a_0 \in S$, $r \in \mathbb{Z}$.

$\rightsquigarrow$ *Define* $\varphi : (\sigma_{a_0}^{\leq r-1} X)[a_0] \to \sigma_{a_0}^{\geq r} X \in \mathbf{MCh}^{\mathsf{b}}_S(\mathcal{A})$ *by*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & X^{\alpha+(r-2)a_0} & \xrightarrow{d_{a_0}} & X^{\alpha+(r-1)a_0} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & \downarrow{\scriptstyle \varphi^{\alpha+(r-1)a_0}} & & \downarrow{\scriptstyle \varphi^{\alpha+ra_0}=d_{a_0}} & & \downarrow{\scriptstyle \varphi^{\alpha+(r+1)a_0}} & & \\ \cdots \longrightarrow & 0 & \longrightarrow & X^{\alpha+ra_0} & \xrightarrow[d_{a_0}]{} & X^{\alpha+(r+1)a_0} & \xrightarrow[d_{a_0}]{} & \cdots \end{array} .$$

Then, for the induced morphism

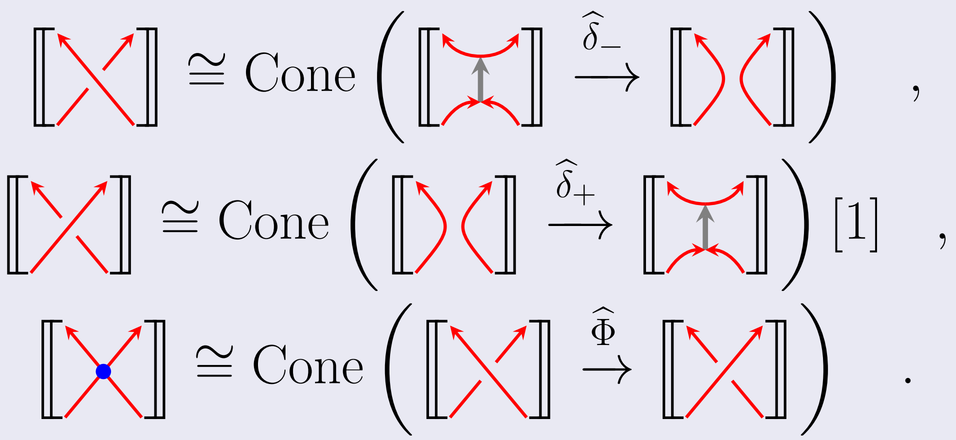
$$\widehat{\varphi} : \mathrm{Tot}((\sigma_{a_0}^{\leq r-1} X)[a_0]^\bullet) \to \mathrm{Tot}(\sigma_{a_0}^{\geq r} X^\bullet) \quad ,$$

we have an isomorphism $\mathrm{Tot}(X^\bullet) \cong \mathrm{Cone}(\widehat{\varphi})$.

Fundamental cofiber sequences

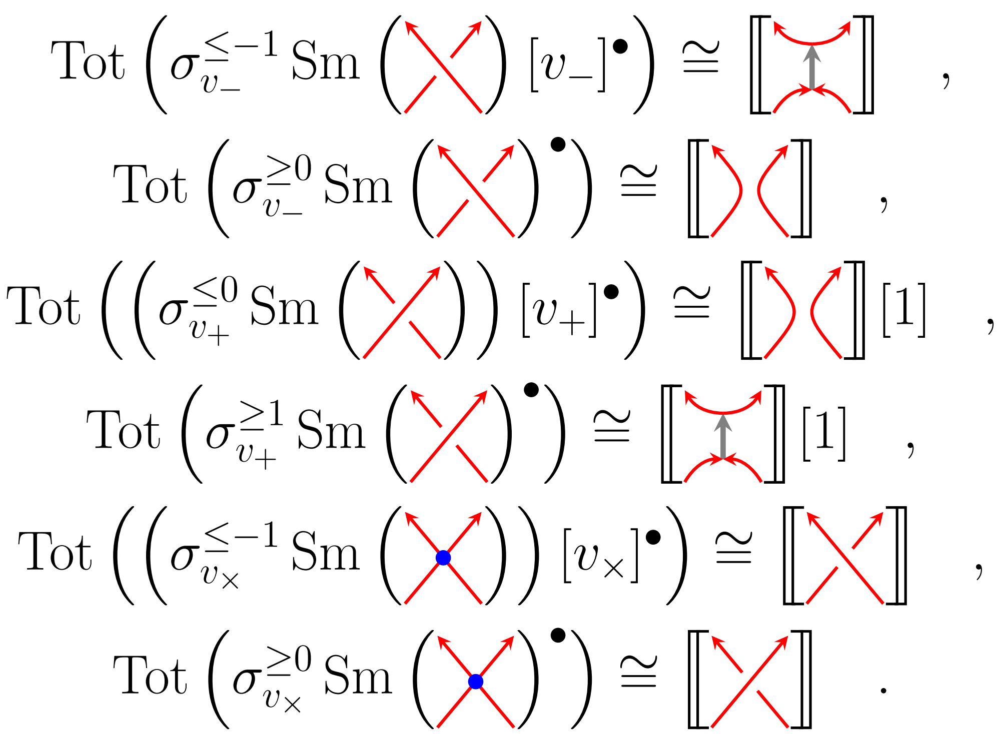
Proposition 2.3

For every **singular link-like graph** D*, there are isomorphisms*



Proof

By direct computations, we obtain



$\rightsquigarrow$ we get the result by Proposition 2.2. $\square$

Corollary 2.4

For every **ordinary link diagram** D, $[\![D]\!]$ *agrees with the one defined in [Bar-Natan, 2005].*

Applying TQFTs

Recall a 2-**dim. TQFT** is nothing but a **Frobenius algebra**.

Fact

For $h, t \in k$, endow a Frobenius algebra structure on $C_{h,t} = k[x]/(x^2 - hx - t)$ by

$$\Delta(1) = 1 \otimes x + x \otimes 1 - h1 \otimes 1 \ , \quad \Delta(x) = x \otimes x + t1 \otimes 1$$
$$\varepsilon(1) = 0 \ , \quad \varepsilon(x) = 1 \quad .$$

Then, it gives rise to a k-linear functor

$$Z_{h,t} : \mathcal{C}ob_2^{\ell}(\varnothing, \varnothing) \to \mathbf{Mod}_k \quad .$$

Example

For every **link diagram** D,

$$HZ_{0,0}[\![D]\!] \cong Kh(D) \, , \; HZ_{1,0}[\![D]\!] \cong BN(D) \, , \; HZ_{0,1}[\![D]\!] \cong Lee(D)$$

Remark

In the case $h = t = 0$, $C_{0,0}$ is **graded** so that

$$\deg 1 = 1 \ , \quad \deg x = -1 \quad .$$

$\rightsquigarrow$ The TQFT $Z_{0,0}$ **respects gradings**.

cf. the Euler grading on $\mathcal{C}ob_2^{\ell}(\varnothing, \varnothing)$.

$\rightsquigarrow$ **Second grading on** Kh, called the q-**grading**.

Invariance

Theorem 2.5 (Ito, Y. 2020)

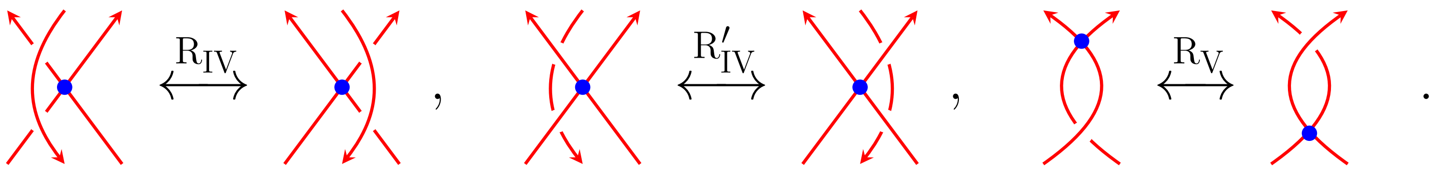
The universal Khovanov complex $[\![D]\!]$ is **invariant** *under the* **moves of singular link diagrams** *up to chain homotopy equivalences.*

$\rightsquigarrow$ Applying $Z_{h,t}$, we get extensions of

- Khovanov homology,
- Lee homology, and
- Bar-Natan homology

to singular links.

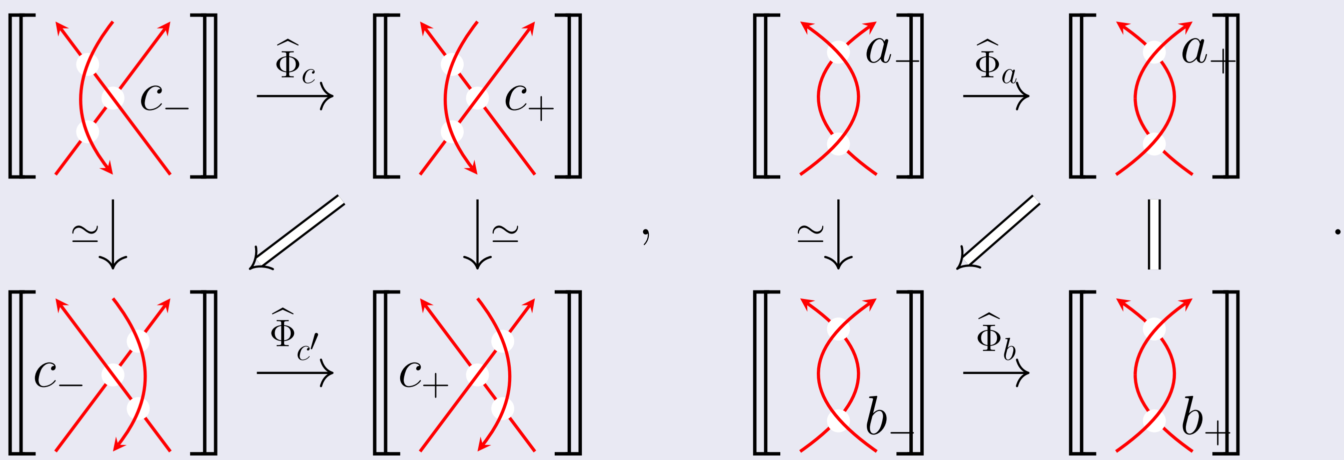
Approach: elementary moves of double points:



Since invariance under **Reidemeister moves** is known, the result essentially follows from Proposition 2.3 and the following.

Proposition 2.6

The **genus-one morphism** $\widehat{\Phi}$ *is* **invariant** *under the moves above; i.e. there are homotopy commutative squares*



The first Vassiliev derivative

Overview

Throughout the section, we fix

G: a **singular link-like graph** with a **unique double point**.

Problem

Compute $Kh(G)$.

Approach

- Construct a complex $[\![G]\!]_{\mathsf{crx}}$ in $\mathcal{C}ob_2^{\ell}(\varnothing, \varnothing)$, called the **crux complex**;

  $\rightsquigarrow C_{\mathsf{crx}}(G) := Z_{0,0}[\![G]\!]_{\mathsf{crx}}$.
- Define $\Xi : [\![G]\!]_{\mathsf{crx}}[2] \to [\![G]\!]_{\mathsf{crx}}[-2]$.
- Show $[\![G]\!] \simeq \mathrm{Cone}(\Xi)$.

  $\rightsquigarrow Kh(G) \cong H^* \mathrm{Cone}(Z_{0,0}(\Xi))$.

Remark

- Ξ has an **explicit description**.
- $[\![G]\!]_{\mathsf{crx}}$ is at least **four times smaller** then $[\![G]\!]$.

  $\rightsquigarrow$ Reduce the size of complexes computing $[\![G]\!]$.

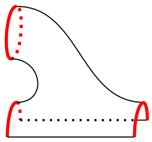
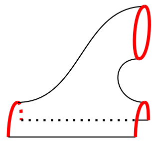
Remark

Regarding **higher derivatives** of Kh, for a general G with exactly r **double points**, we have a **spectral sequence**

$$E_1 \cong (1^{st}\text{-derivatives of } Kh) \Rightarrow Kh(G) \quad .$$

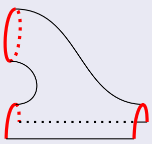
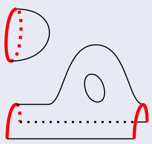
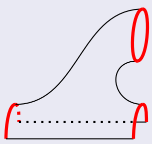
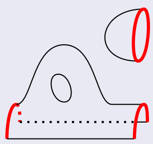
Twisted action

Recall in the category of **cobordisms**, the interval I is a **module** and a **comodule** over the circle S^1:

$\mu :=$  $: I \otimes S^1 \to I$, $\Delta :=$  $: I \to I \otimes S^1$

Proposition 3.1

The following also give other S^1**-module/comodule structures** *on* I *up to Bar-Natan's* **4Tu-relation***:*

$\widetilde{\mu} :=$  $-$  , $\widetilde{\Delta} :=$  $-$  .

We call them **twisted action/coaction** *of* S^1 *on* I*.*

Example

$C_{0,0} = k[x]/(x^2)$: The Frobenius algebra **for Khovanov homology**.

$\rightsquigarrow$ The **twisted action/coaction** are

$$\begin{array}{rcl} \widetilde{\mu} : C_{0,0} \otimes C_{0,0} & \to & C_{0,0} \\ a \otimes 1 & \mapsto & a \\ a \otimes x & \mapsto & -ax \end{array} , \quad \begin{array}{rcl} \widetilde{\Delta} : C_{0,0} & \mapsto & C_{0,0} \otimes C_{0,0} \\ 1 & \mapsto & 1 \otimes x - x \otimes 1 \\ x & \mapsto & x \otimes x \end{array} .$$

Remark

In the situation above, $\widetilde{\mu}$ and $\widetilde{\Delta}$ are homogeneous with respect to the **Euler degrees** and

$$\deg \widetilde{\mu} = \deg \widetilde{\Delta} = \deg \mu = \deg \Delta = -1 .$$

Crux complexes

Definition

A map $\alpha : c(G) \to \mathbb{Z}$ is called a **G-crux map** if

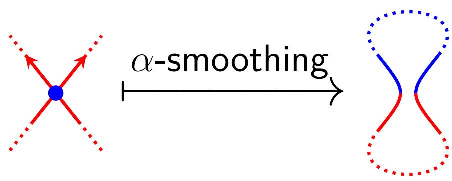
1. α lies in the effective range (with $\alpha(\textbf{double pt.}) := 0$);
2. the edges adjacent to the (unique!) double point lie in the same connected component in $|G_\alpha|$.

Notation

$$|G_\alpha|_{\mathsf{crx}} := \begin{cases} |G_\alpha| & \alpha\text{: } G\text{-crux map,} \\ 0 & \text{otherwise.} \end{cases}$$

Twisted arcs

The **"upper half"** of the component encircling the double point:

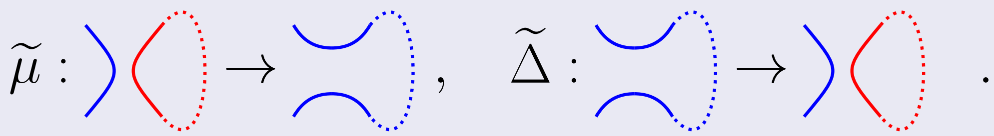


Definition

Define a $c(G)$-fold complex $\mathrm{Crx}(G)^\bullet$ by

$$\mathrm{Crx}(G)^\alpha := |G_\alpha|_{\mathsf{crx}}$$

with the differentials given in the same way as $\mathrm{Sm}(G)^\bullet$ with **twisted actions/coactions** on **twisted arcs**:



Crux complexes

In order to verify $\mathrm{Crx}(G)^\bullet$ is actually an $c(G)$-fold complex, we have to check the following.

Lemma 3.2

For $v \neq w \in c(G)$, the following commutes:

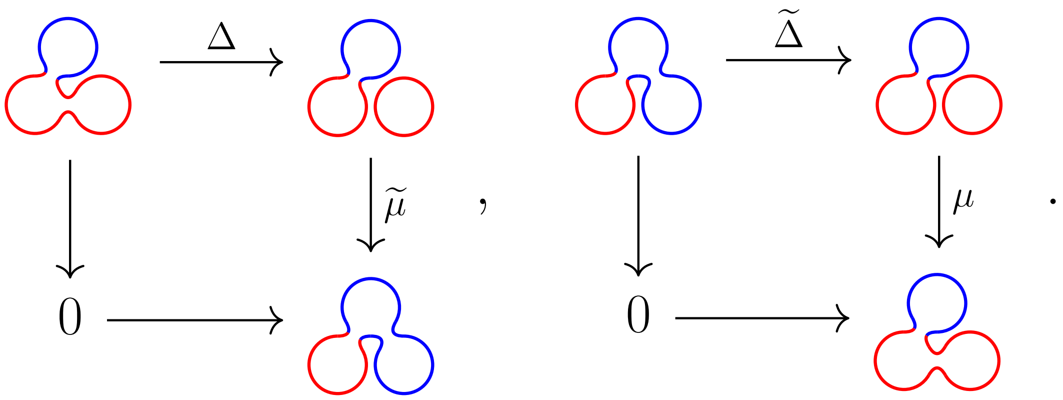
$$\begin{array}{ccc} |G_\alpha|_{\text{crx}} & \xrightarrow{d_v} & |G_{\alpha+v}|_{\text{crx}} \\ {\scriptstyle d_w}\downarrow & & \downarrow{\scriptstyle d_w} \\ |G_{\alpha+w}|_{\text{crx}} & \xrightarrow{d_w} & |G_{\alpha+v+w}|_{\text{crx}} \end{array} .$$

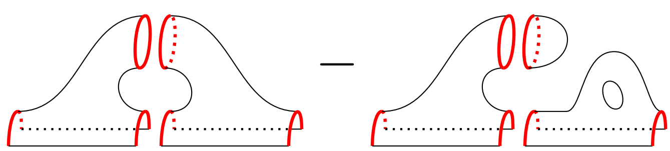
Proof

CASE $d_v, d_w \in \{\mu, \Delta\}$: same as $\mathrm{Sm}(-)^\bullet$.

CASE $d_v, d_w \in \{\mu, \widetilde{\mu}\}$ or $d_v, d_w \in \{\Delta, \widetilde{\Delta}\}$: The result follows from the **(co)associativity** of $\widetilde{\mu}$ and $\widetilde{\Delta}$.

The other non-trivial cases: It remains to discuss



They are verified e.g. as  $= 0$

∎

Crux complexes

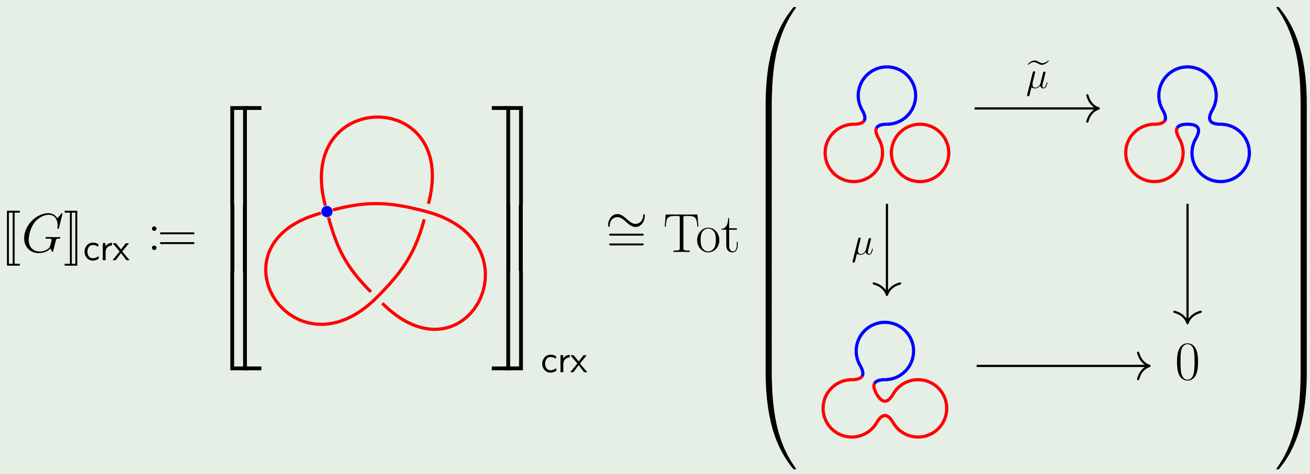
Definition

The complex $[\![G]\!]_{\mathsf{crx}} := \mathrm{Tot}(\mathrm{Crx}(G)^{\bullet})$ is called the **crux complex** of G.

In particular, write

$$C_{\mathsf{crx}}(G) := Z_{0,0}[\![G]\!]_{\mathsf{crx}} \quad .$$

Example



$\rightsquigarrow$ We obtain $H^*C_{\mathsf{crx}}(G) = 0$ except

$$H^{-2}C_{\mathsf{crx}}(G) \cong \langle x \otimes x \rangle \cong \mathbb{Z} \ ,$$

$$H^{-1}C_{\mathsf{crx}}(G) \cong C_{0,0}^{\oplus 2} / \big\langle (1,-1), (x,x), (x,-x) \big\rangle$$
$$\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \ .$$

Remark

In the example above, the summands in $H^*C_{\mathsf{crx}}(G)$ are homogeneous with respect to the q-degree.

Absolute exact sequences

Strategy

To relate $[\![G]\!]_{\mathsf{crx}}$ with $[\![G]\!]$, we want to place them in a single **exact sequence**.

YET, $\mathcal{C}ob^{\ell}_{2}(\varnothing, \varnothing)$ is **not abelian**.

$\rightsquigarrow$ What is **exactness**?

Definition

A sequence

$$\cdots \xrightarrow{f^{i-1}} X^i \xrightarrow{f^i} X^{i+1} \xrightarrow{f^{i+1}} \cdots$$

in a (pre-)additive category $\mathcal{A}$ is said to be **absolutely exact** if the following conditions are satisfied:

1. $f^{i+1}f^i = 0$ for every $i \in \mathbb{Z}$;
2. it is contractible as a chain complex.

Remark

TFAE:

- $\{X^i, f^i\}$ is absolutely exact;
- For every functor $F : \mathcal{A} \to \mathcal{V}$ with $\mathcal{V}$ abelian, $\{FX^i, Ff^i\}$ is exact.

Example

1. Absolute exact sequence of length 1 = zero object
2. Absolute exact sequence of length 2 = isomorphism
3. Absolute exact sequence of length 3 = split short exact sequence

Key exact sequence

Proposition 3.3 (cf. RI-moves by [Bar-Natan, 2005])

The following are **split short exact sequences***:*

$$0 \to I \xrightarrow{\Delta} I \otimes S^1 \xrightarrow{\widetilde{\mu}} I \to 0 \quad ,$$

$$0 \to I \xrightarrow{\widetilde{\Delta}} I \otimes S^1 \xrightarrow{\mu} I \to \quad .$$

Proof

First, observe that the S-relation implies

$$\mu \circ (I \otimes \eta) = \widetilde{\mu} \circ (I \otimes \eta) = (I \otimes \varepsilon) \circ \Delta = (I \otimes \varepsilon) \circ \widetilde{\Delta} = \mathrm{id}_I \quad .$$

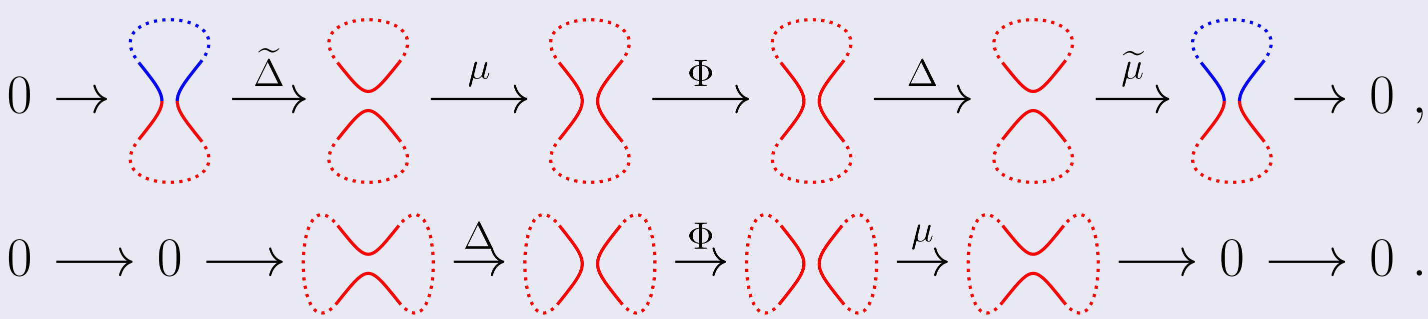
In addition, the $4Tu$-relation yields

$$\Delta \circ (I \otimes \varepsilon) + (I \otimes \eta) \circ \widetilde{\mu} = \widetilde{\Delta} \circ (I \otimes \varepsilon) + (I \otimes \eta) \circ \mu = \mathrm{id}_{I \otimes S^1} \quad .$$

□

Corollary 3.4

The following are absolutely exact sequences:



Proof

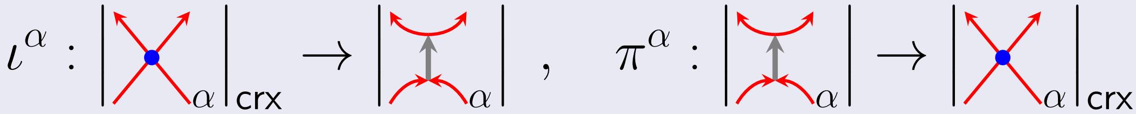
The first one is obvious since $\Phi = 0$ in this case.

The second follows from the observation $\Phi = \widetilde{\Delta}\widetilde{\mu}$. □

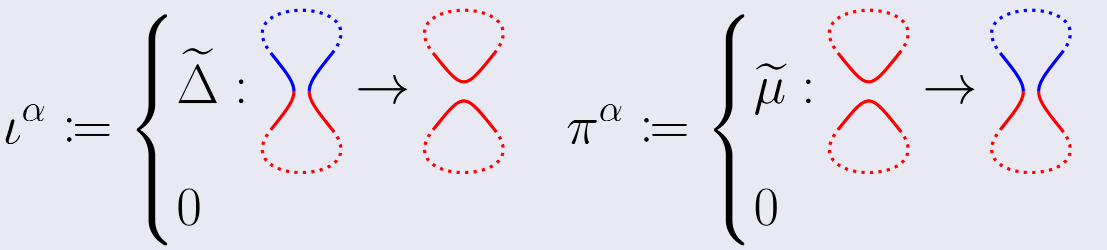
Key exact sequence

Definition

For $\alpha : c(G) \to \mathbb{Z}$, define morphisms



by

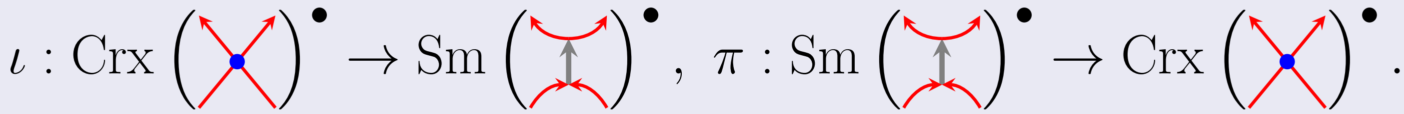


α: G-crux map,

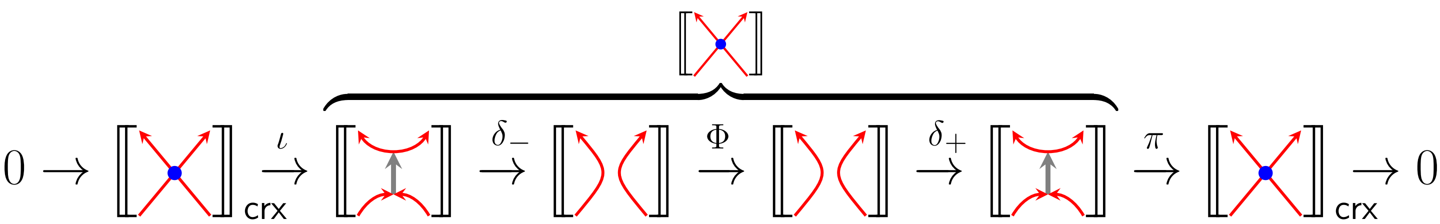
otherwise,

Lemma 3.5

The families $\iota = \{\iota^\alpha\}_\alpha$ and $\pi = \{\pi^\alpha\}_\alpha$ form the following morphisms of $c(G)$-fold complexes respectively:



We obtain a sequence of complexes



which is degreewisely **absolutely exact**.

$\rightsquigarrow$ The **"bulk"** part should be determined by the **"edge"** part.

Generalized 9-lemma

Proposition 3.6 (Generalized 9-lemma (absolute version))

If $X^{i,j}$ is a bounded bicomplex which is **absolutely exact** *on each* **fixed vertical degree**, *then $\forall p+1<q$ there is a morphism*

$$\Xi : \mathrm{Tot}(\sigma_{\mathsf{H}}^{\geq q} X) \to \mathrm{Tot}(\sigma_{\mathsf{H}}^{\leq p} X)[1]$$

of chain complexes such that

$$\mathrm{Cone}(\Xi) \simeq \mathrm{Tot}(\sigma_{\mathsf{H}}^{\geq p+1} \sigma_{\mathsf{H}}^{\leq q-1} X) \quad .$$

Sketch

1. We have an isomorphism

$$\mathrm{Tot}(X) \cong \mathrm{Cone}\left(\mathrm{Tot}(\sigma_{\mathsf{H}}^{\leq q-1} X)[1] \xrightarrow{\widehat{\varphi}_q} \mathrm{Tot}(\sigma_{\mathsf{H}}^{\geq q} X)\right)$$

with LHS **contractible**.
$\rightsquigarrow$ $\widehat{\varphi}_q$ is a homotopy equivalence.

2. We also have an isomorphism

$$\mathrm{Tot}(\sigma_{\mathsf{H}}^{\leq q-1}) \cong \mathrm{Cone}\left(\mathrm{Tot}(\sigma_{\mathsf{H}}^{\leq p} X)[1] \xrightarrow{\widehat{\varphi}_p} \mathrm{Tot}(\sigma_{\mathsf{H}}^{\geq p+1} \sigma_{\mathsf{H}}^{\leq q-1} X)\right)$$

3. Combining 1 and 2, we obtain the following **distinguished triangle** in the homotopy category:

$$\mathrm{Tot}(\sigma_{\mathsf{H}}^{\leq p} X)[1] \xrightarrow{\widehat{\varphi}_p} \mathrm{Tot}(\sigma_{\mathsf{H}}^{\geq p+1} \sigma_{\mathsf{H}}^{\leq q-1} X) \xrightarrow{\widehat{\varphi}_q \circ i} \mathrm{Tot}(\sigma_{\mathsf{H}}^{\geq q} X)[-1]$$

$\rightsquigarrow$ required Ξ obtained by **rotating** the triangle. □

Proof of Main Theorem II

Theorem 3.7

G: a singular link-like graph with a unique double point.

$\rightsquigarrow$ *There is a morphism $\Xi : [\![G]\!]_{\mathsf{crx}}[2] \to [\![G]\!]_{\mathsf{crx}}[-2]$ such that*

$$[\![G]\!] \cong \mathrm{Cone}(\Xi) \quad .$$

Applying the TQFT $Z_{0,0}$, we finally obtain Main Theorem II.

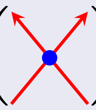
Main Theorem II

D: a **singular link diagram** *with* **exactly one double point***.*

$\rightsquigarrow$ $\exists$ *chain complex $C^{*,\star}_{\mathsf{crx}}(D)$ together with a (graded) endomorphism*

$$\Xi : C^{*-2,\star-2}_{\mathsf{crx}}(D) \to C^{*+2,\star+4}_{\mathsf{crx}}(D)$$

such that

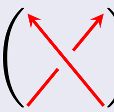
C_{Kh} () $\simeq \mathrm{Cone}(\Xi) \quad .$

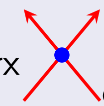
Remark

Main Theorem II **categorifies** the equation

$$V_L(e^{\frac{2\pi}{3}\sqrt{-1}}) = (-1)^{\#L-1} \quad .$$

$\because$ Taking the Euler characteristics in Main Theorem II (cf. $q = -t^{1/2}$),

$\chi\left(Kh\left(\right.\right.$  $\left.\left.\right)\right) - \chi\left(Kh\left(\right.\right.$  $\left.\left.\right)\right)$

$= (q^4 - q^{-2})\chi\left(H^* C_{\mathsf{crx}}\right.$  $\left.{}_{\mathsf{crx}}\right) \quad ,$

which belongs to the ideal generated by $(1 - t^3)$.

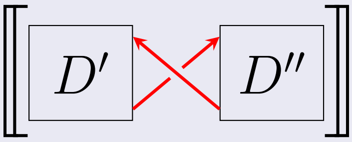
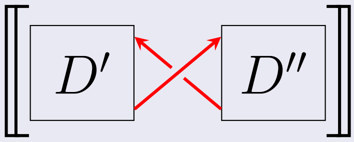
$\rightsquigarrow$ Crossing change does not affect the value at $t = \sqrt[3]{1}$.

Application

Reducible crossing

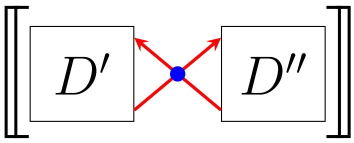
Theorem 4.1

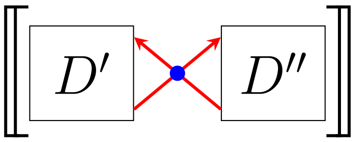
The **genus-one** *morphism* $\widehat{\Phi}$ *is a homotopy-equivalence for* **reducible** *(aka.* **nugatory***) crossing; i.e.*

$\widehat{\Phi}$:  $\xrightarrow{\sim}$  .

Proof

By the **categorified Vassiliev skein relation**,

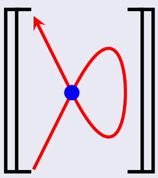
$\widehat{\Phi}$: homotopy equivalence $\Longleftrightarrow$  $\simeq 0$.

On the other hand, we have  $=_{\text{crx}} 0$.

($\because$ no crux map) □

Corollary 4.2

The universal Khovanov complex satisfies a **homotopy FI relation***:*

 $\simeq 0$.

Homology of twist knots

Theorem 4.3 (Y. arXiv:2007.15867)

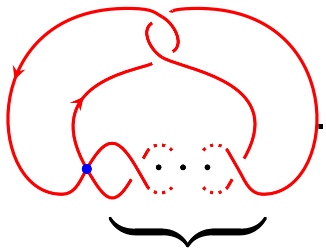
There is a homotopy equivalence

$$\left[\!\left[G \right]\!\right] \simeq \begin{cases} [\![\text{unknot}]\!] \oplus \bigoplus_{i=1}^{r/2} C(2i-1) & \text{if } r \text{ is even,} \\ [\![\text{trefoil}]\!] \oplus \bigoplus_{i=1}^{(r-1)/2} C(2i) & \text{if } r \text{ is odd,} \end{cases}$$

here $C(k)$ is the complex of the form

$$\cdots \to 0 \to S^1 \xrightarrow{\mu\Delta} S^1 \to \overset{-k-(-1)^k}{\underset{\smile}{0}} \to S^1 \xrightarrow{\mu\Delta} S^1 \to 0 \to \cdots \quad .$$

Sketch

1. Induction on r using $G(r) :=$  .

2. Observe $\Xi = 0$ on $G(r)$ for the degree reason.
3. $G(r)$-crux map α values -1 on the horizontal twists.
   $\rightsquigarrow$ $[\![G(r)]\!]_{\text{crx}}$ **does not depend on** r up to shifts.
4. Compute $[\![G(r)]\!]_{\text{crx}}$.
   $\rightsquigarrow$ the half of $C(k)$ appears.
5. Verify the splitting in each induction step.

□

[Bar-Natan, 2005] Bar-Natan, D. (2005).

Khovanov's homology for tangles and cobordisms.

Geometry & Topology, 9(3):1443–1499.

[Birman, 1993] Birman, J. S. (1993).

New points of view in knot theory.

American Mathematical Society. Bulletin. New Series, 28(2):253–287.

[Birman and Lin, 1993] Birman, J. S. and Lin, X.-S. (1993).

Knot polynomials and Vassiliev's invariants.

Inventiones Mathematicae, 111(2):225–270.

[Budney et al., 2017] Budney, R., Conant, J., Koytcheff, R., and Sinha, D. (2017).

Embedding calculus knot invariants are of finite type.

Algebraic & Geometric Topology, 17(3):1701–1742.

[Hedden and Watson, 2018] Hedden, M. and Watson, L. (2018).

On the geography and botany of knot Floer homology.

Selecta Mathematica. New Series, 24(2):997–1037.

[Ito and Yoshida, 2020] Ito, N. and Yoshida, J. (2020).

Universal Khovanov homology for singular tangles and a categorified Vassiliev skein relation.

arXiv:2005.12664.

[Khovanov, 2000] Khovanov, M. (2000).

A categorification of the Jones polynomial.

Duke Mathematical Journal, 101(3):359–426.

Reference II

[Kronheimer and Mrowka, 2011] Kronheimer, P. B. and Mrowka, T. S. (2011).

Khovanov homology is an unknot-detector.

Publications Mathématiques. Institut de Hautes Études Scientifiques, 113:97–208.

[Lee, 2005] Lee, E. S. (2005).

An endomorphism of the Khovanov invariant.

Advances in Mathematics, 197(2):554–586.

[Rasmussen, 2010] Rasmussen, J. A. (2010).

Khovanov homology and the slice genus.

Inventiones Mathematicae, 182(2):419–447.

[Vassiliev, 1990] Vassiliev, V. A. (1990).

Cohomology of knot spaces.

In *Theory of singularities and its applications*, volume 1 of *Advances in Soviet Mathematics*, pages 23–69. American Mathematical Society, Providence, RI.

[Volić, 2006] Volić, I. (2006).

Finite type knot invariants and the calculus of functors.

Compositio Mathematica, 142(1):222–250.

[Yoshida, 2020] Yoshida, J. (2020).

Decomposition of the first Vassiliev derivative of Khovanov homology and its application.

arXiv:2007.15867.