The intersection polynomials of a virtual knot.

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joint work with R. Higa (Kobe Univ.), T. Nakamura (Yamanashi Uinv.)
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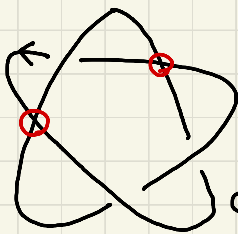
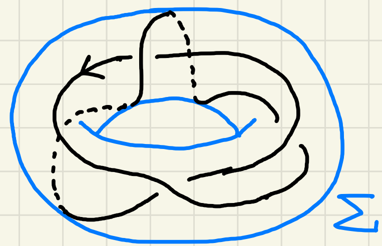
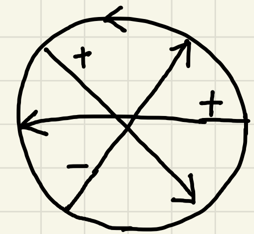
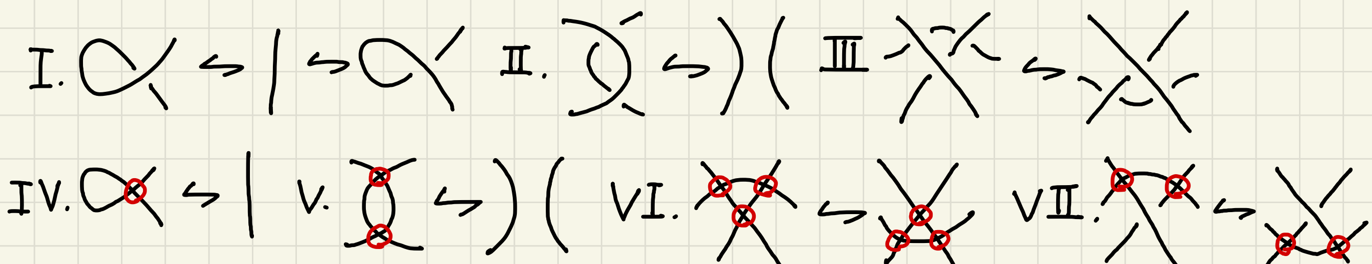
§1. What is a virtual knot?

①  $\subset \mathbb{R}^2$

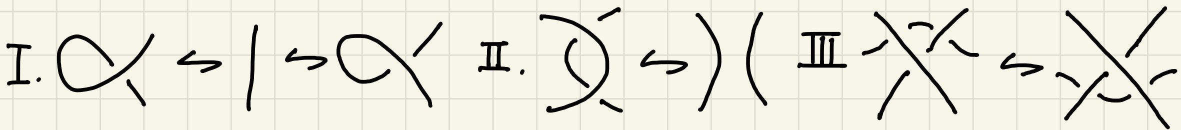
② 

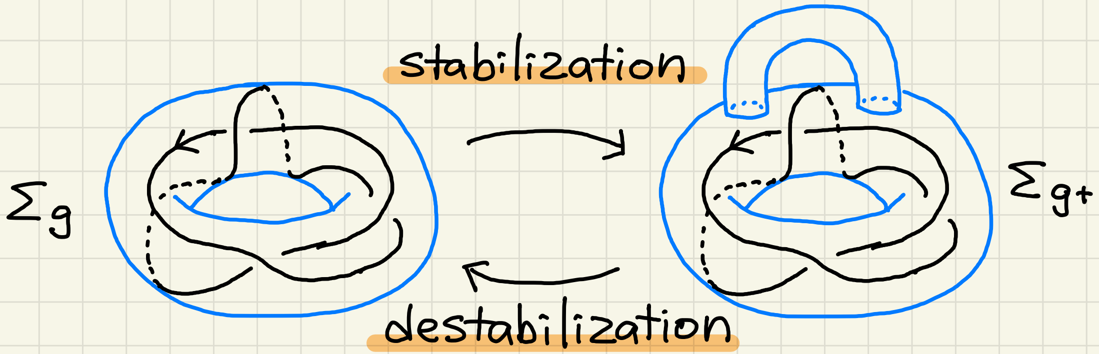
③ 

① A diagram $\subset \mathbb{R}^2$ with real/virtual crossings

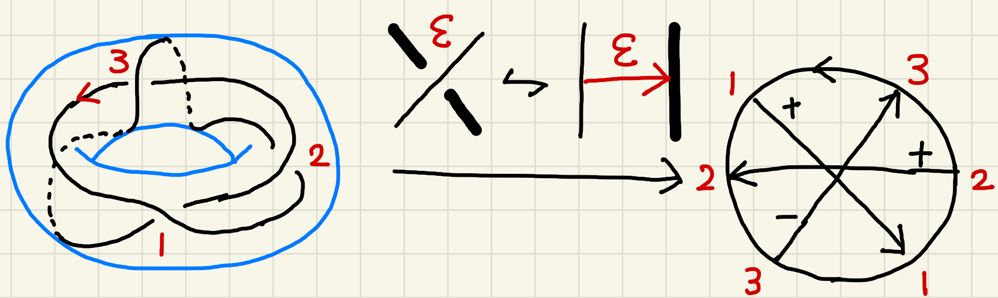


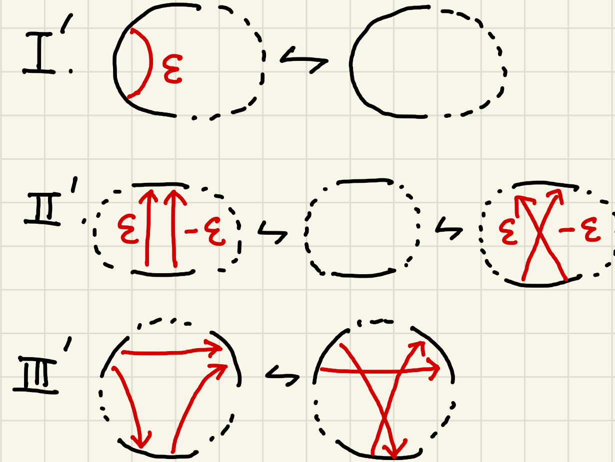
② A diagram $\subset \Sigma$ (a conn. ori. closed surface) with (real) crossings





③ A Gauss diagram





$$\left\{\begin{array}{l}\text{diag.} \subset \mathbb{R}^2 \text{ with} \\ \text{real/virtual} \\ \text{crossings}\end{array}\right\} \Big/ \text{I–VII.} \quad \overset{1:1}{\longleftrightarrow} \quad \left\{\begin{array}{l}\text{diag.} \subset \Sigma \text{ with} \\ \text{(real) crossings}\end{array}\right\} \Big/ \begin{array}{l}\text{I–III} \\ \text{(de)stab.}\end{array} \quad \overset{1:1}{\longleftrightarrow} \quad \{\text{Gauss diag.}\} \big/ \text{I}'\text{–III}'.$$

$\rightsquigarrow$ virtual knots

Rem. (i) $D, D' \subset \mathbb{R}^2$ with real crossings only (classical diag.)

$$D \overset{\text{I–VII}}{\sim} D' \Longleftrightarrow D \overset{\text{I–III.}}{\sim} D'$$

(ii) $sg(K) := \min\{g \mid (D, \Sigma_g) \text{ presents } K\}$. supporting genus

$$(D, \Sigma_{sg(K)}) \underset{\text{(de)stab.}}{\overset{\text{I–III.}}{\sim}} (D', \Sigma_{sg(K)}) \Longleftrightarrow (D, \Sigma_{sg(K)}) \overset{\text{I–III}}{\sim} (D', \Sigma_{sg(K)})$$

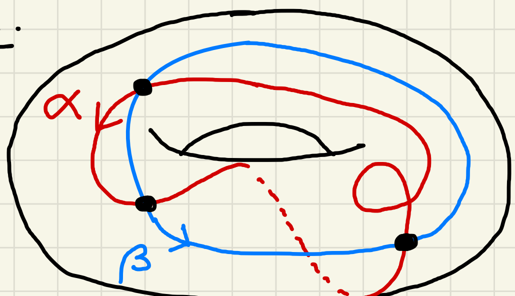
$\rightsquigarrow$ {virtual knots} $\supset$ {classical knots}

§2. The writhe polynomial.

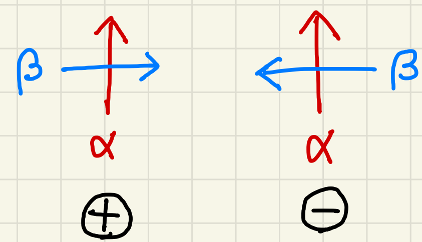
Invariants of virtual knots $\left\{ \begin{array}{l} \text{① generalizations of invariants of classical knots (knot group, Jones poly, } \cdots) \\ \text{② vanish for any classical knots (Sawollek poly, odd writhe, writhe poly } \cdots) \end{array} \right.$

◎ $\alpha, \beta \subset \Sigma \rightsquigarrow \alpha \cdot \beta \in \mathbb{Z}$ (the intersection number).

Ex.



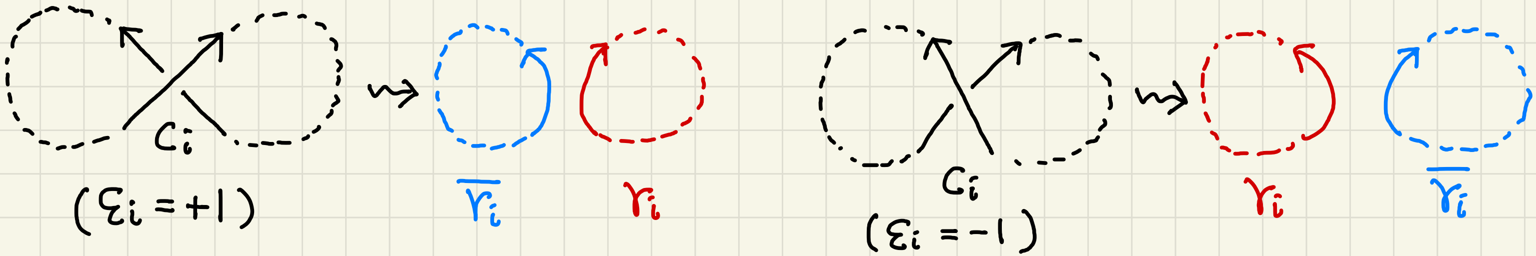
$$\begin{cases} \alpha \cdot \beta = 1 \\ \beta \cdot \alpha = -1 \end{cases}$$



Rem. $\forall \alpha, \beta \in H_1(\Sigma)$, $\alpha \cdot \alpha = 0$. $\alpha \cdot \beta = -\beta \cdot \alpha$

◎ $D \subset \Sigma$: a diag. of a virtual knot K

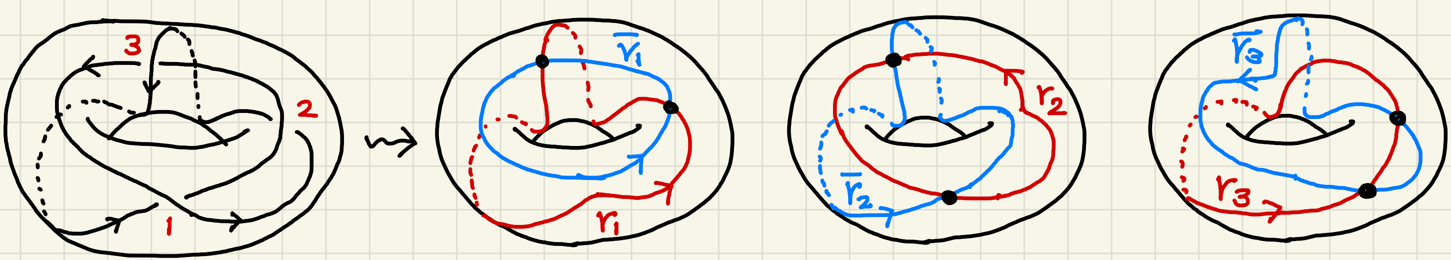
$c_1, \cdots, c_n$: the crossings of D.



$\rightsquigarrow$ $\gamma_1, \cdots, \gamma_n \subset \Sigma$ (from the over-crossings to the under)

$\overline{\gamma_1}, \cdots, \overline{\gamma_n} \subset \Sigma$ (from the under-crossings to the over)

Ex.



$\gamma_1 \cdot \overline{\gamma_1} = 2$ $\qquad$ $\gamma_2 \cdot \overline{\gamma_2} = -2$ $\qquad$ $\gamma_3 \cdot \overline{\gamma_3} = 0$

Def. $W_K(t) = \sum_{i=1}^{n} \varepsilon_i \left(t^{\gamma_i \cdot \overline{\gamma_i}} - 1\right)$: the writhe polynomial of K

Ex. $W_K(t) = +(t^2-1)+(t^{-2}-1)-(t^0-1) = t^2-2+t^{-2}$.

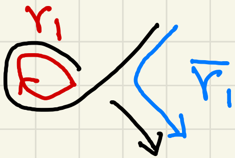
◎ Well-definedness

$(D,\Sigma_g) = (D_1,\Sigma_{g_1}), (D_2,\Sigma_{g_2}), \cdots, (D_s,\Sigma_{g_s}) = (D',\Sigma_{g'})$

s.t. $(D_{i+1}, \Sigma_{g_{i+1}})$ is obtained from (D_i, Σ_{g_i}) by

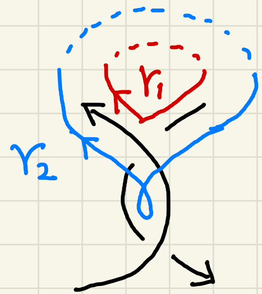
(0) a homeo. of Σ_{g_i}, (1) a (de)stabilization, (2) a Reidemeister move.

I.



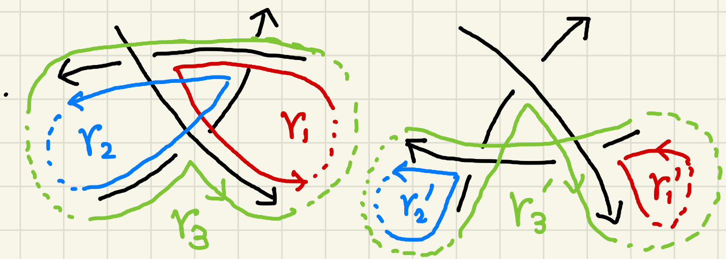
$\gamma_1 \cdot \overline{\gamma_1} = 0$

II.



$\gamma_1 = \gamma_2 \Rightarrow \gamma_1 \cdot \overline{\gamma_1} = \gamma_2 \cdot \overline{\gamma_2}$

III.



$\gamma_i = \gamma_i' \Rightarrow \gamma_i \cdot \overline{\gamma_i} = \gamma_i' \cdot \overline{\gamma_i}'$

$(i=1,2,3)$ ∎

§3. The intersection polynomials.

$D \subset \Sigma$: a diag. of K with crossings $c_1, \cdots, c_n$

$\gamma_i, \overline{\gamma_i} \subset \Sigma$: the cycles at c_i $(1 \le i \le n)$, ε_i : the sign of c_i

$$f_{00}(D;t) = \sum_{1 \le i,j \le n} \varepsilon_i \varepsilon_j (t^{\gamma_i \cdot \gamma_j} - 1), \quad f_{01}(D;t) = \sum_{1 \le i,j \le n} \varepsilon_i \varepsilon_j (t^{\gamma_i \cdot \overline{\gamma_j}} - 1)$$

$$f_{10}(D;t) = \sum_{1 \le i,j \le n} \varepsilon_i \varepsilon_j (t^{\overline{\gamma_i} \cdot \gamma_j} - 1), \quad f_{11}(D;t) = \sum_{1 \le i,j \le n} \varepsilon_i \varepsilon_j (t^{\overline{\gamma_i} \cdot \overline{\gamma_j}} - 1).$$

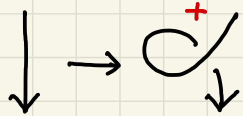
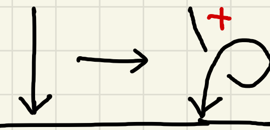
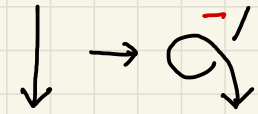
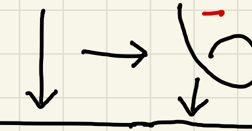
Lem. $f_{10}(D;t) = f_{01}(D;t^{-1})$

$\because$ $\overline{\gamma_i} \cdot \gamma_j = -\gamma_j \cdot \overline{\gamma_i}$ ■

Lem. $D \to D'$: a Reidemeister move II or III $\Rightarrow f_{pq}(D) = f_{pq}(D')$ $\forall p,q$

$\because$ The proof is the same as that for $W_K(t)$. ■

Lem.

$D \to D'$				
$f_{01}(D')-f_{01}(D)$	$W_K(t)$	$W_K(t)$	$-W_K(t)$	$-W_K(t)$
$f_{00}(D')-f_{00}(D)$	0	$W_K(t)+W_K(t^{-1})$	$-W_K(t)-W_K(t^{-1})$	0
$f_{11}(D')-f_{11}(D)$	$W_K(t)+W_K(t^{-1})$	0	0	$-W_K(t)-W_K(t^{-1})$

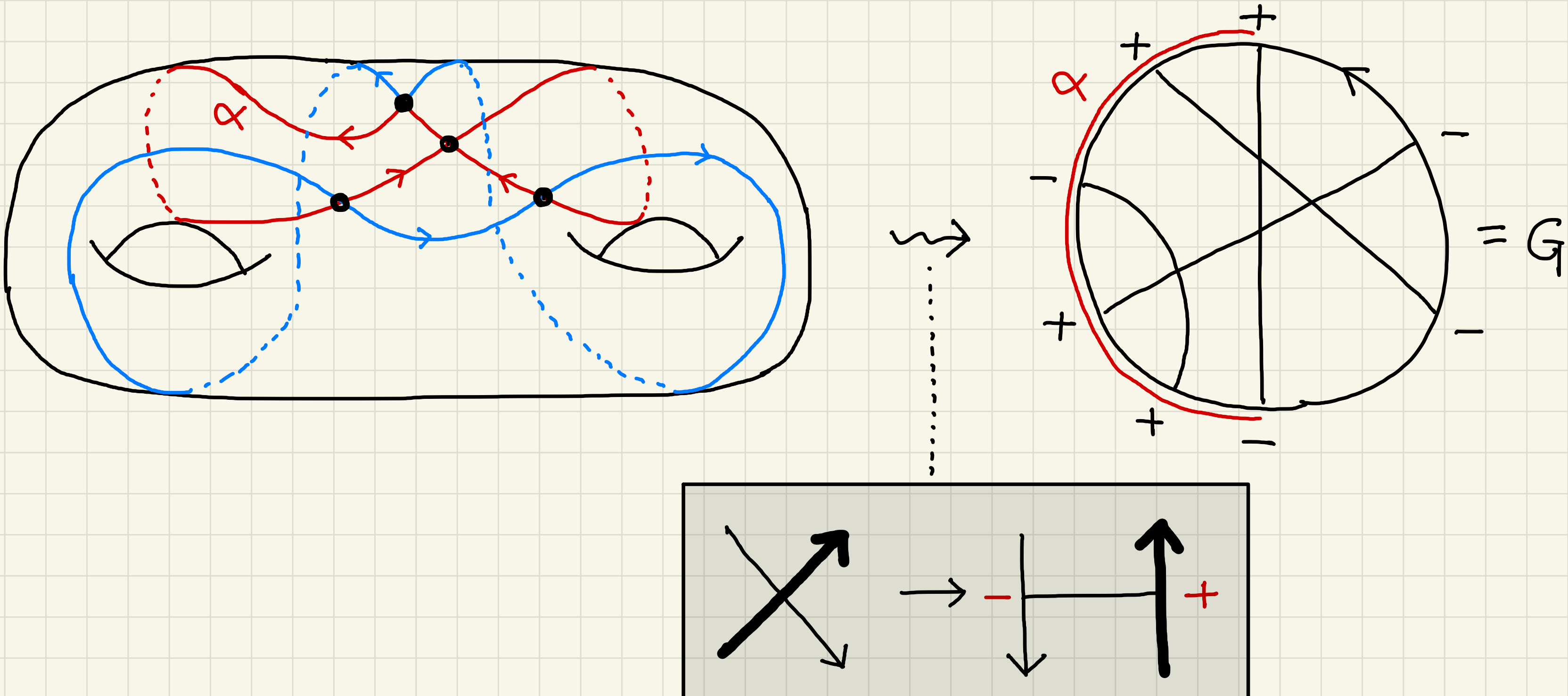
$\omega_D = \sum_{i=1}^{n} \varepsilon_i$: the writhe of D.

Def. $$I_K(t) = \sum_{1\le i,j\le n} \varepsilon_i \varepsilon_j \left(t^{\gamma_i \cdot \overline{\gamma_j}} - 1\right) - \omega_D \cdot W_K(t)$$

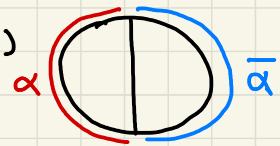
$$II_K(t) = \sum_{1\le i,j\le n} \varepsilon_i \varepsilon_j \left(t^{\gamma_i \cdot \gamma_j} - 1\right) + \sum_{1\le i,j\le n} \varepsilon_i \varepsilon_j \left(t^{\overline{\gamma_i} \cdot \overline{\gamma_j}} - 1\right) - \omega_D \cdot \left(W_K(t) + W_K(t^{-1})\right)$$

: the first and second intersection polynomials of K

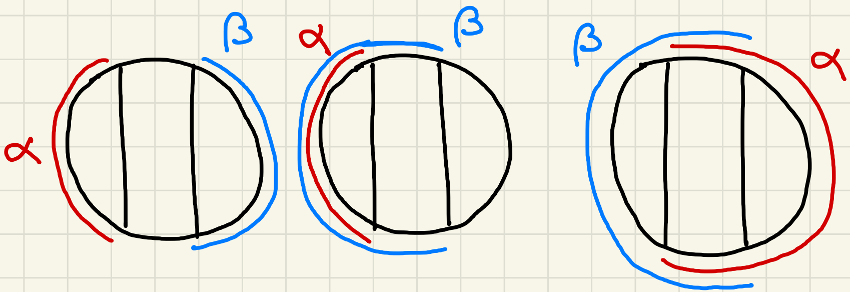
◎ How to calculate $W_K(t)$, $I_K(t)$, $II_K(t)$ from a Gauss diagram.



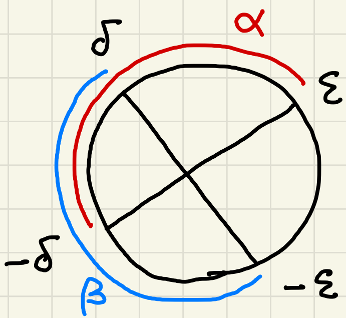
$S(\alpha,\beta)$ = the sum of the signs of endpoints of chords on $\operatorname{int}(\alpha)$ s.t. the other endpoints of the chords lie on $\operatorname{int}(\beta)$

Lem. (i) 

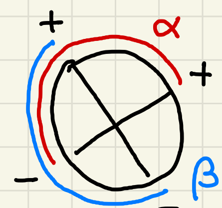
$\Rightarrow \alpha\cdot\bar{\alpha} = S(\alpha,\bar{\alpha})$

(ii) 

$\Rightarrow \alpha\cdot\beta = S(\alpha,\beta)$

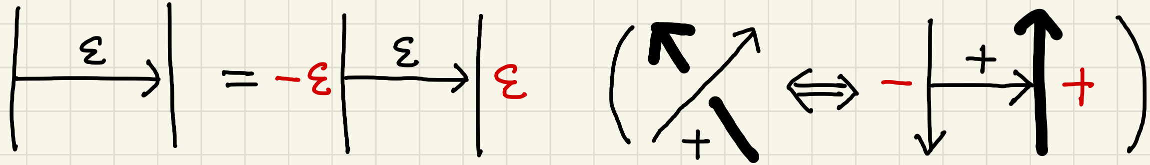
(iii) 

$\Rightarrow \alpha\cdot\beta = S(\alpha,\beta) + \frac{1}{2}(\varepsilon+\delta)$.

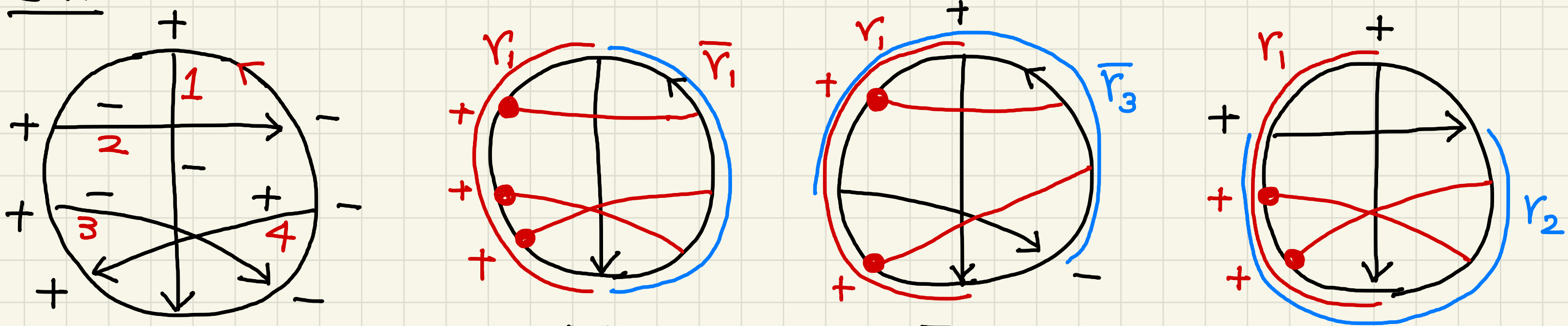
[ex. 

$\alpha\cdot\beta = S(\alpha,\beta) + 1$

$\beta\cdot\alpha = S(\beta,\alpha) - 1$]

Rem. $G \supset$ 

Ex.



$r_1 \cdot \bar{r}_1 = 3 \qquad r_1 \cdot \bar{r}_3 = 2 \qquad r_1 \cdot r_2 = \underbrace{2+1}_{} = 3$

$$W_K(t) = \sum_i \varepsilon_i \left(t^{r_i \cdot \bar{r}_i} - 1\right) = -t^3 + t^2 + 1 - t^{-1}$$

$$\begin{cases} f_{01}(D) = \sum_{i,j} \varepsilon_i \varepsilon_j \left(t^{r_i \cdot \bar{r}_j} - 1\right) = 2t^2 - 2t - 2 + 2t^{-1} \\ f_{00}(D) = \sum_{i,j} \varepsilon_i \varepsilon_j \left(t^{r_i \cdot r_j} - 1\right) = t^3 - t^2 - t^{-2} - t^{-3} \end{cases}$$

$$f_{11}(D) = \sum_{i,j} \varepsilon_i \varepsilon_j \left(t^{\overline{r_i} \cdot \overline{r_j}} - 1\right) = t^2 - t - t^{-1} + t^{-2}$$

$$\Rightarrow \mathrm{I}_K(t) = f_{01}(D) - \omega_D W_K(t) = -2t^3 + 4t^2 - 2t.$$

$$\mathrm{II}_K(t) = f_{00}(D) + f_{11}(D) - \omega_D\left(W_K(t) + W_K(t^{-1})\right)$$

$$= -t^3 + 2t^2 - 3t + 4 - 3t^{-1} + 2t^{-2} - t^{-3}.$$

crossing # of K

◎ We calculated $\mathrm{I}_K(t)$ and $\mathrm{II}_K(t)$ for all K with $c(K) \leq 4$.

Thm. $\exists (K_1, K_2)$ s.t.

(i) $W_{K_1}(t) \neq W_{K_2}(t)$, $\mathrm{I}_{K_1}(t) = \mathrm{I}_{K_2}(t)$, $\mathrm{II}_{K_1}(t) = \mathrm{II}_{K_2}(t)$

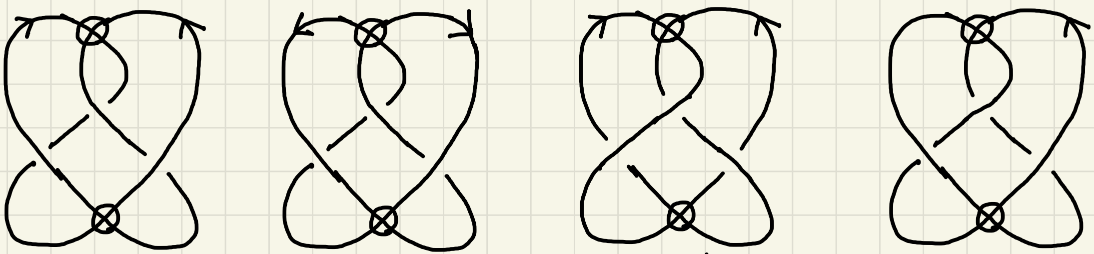
(ii) $W_{K_1}(t) = W_{K_2}(t)$, $\mathrm{I}_{K_1}(t) \neq \mathrm{I}_{K_2}(t)$, $\mathrm{II}_{K_1}(t) = \mathrm{II}_{K_2}(t)$

(iii) $W_{K_1}(t) = W_{K_2}(t)$, $\mathrm{I}_{K_1}(t) = \mathrm{I}_{K_2}(t)$, $\mathrm{II}_{K_1}(t) \neq \mathrm{II}_{K_2}(t)$

§4. Properties of $\mathrm{I}_K(t)$ and $\mathrm{II}_K(t)$.

Lem. (i) K: classical $\Longrightarrow$ $\mathrm{I}_K(t) = \mathrm{II}_K(t) = 0$.

(ii) $\mathrm{II}_K(t)$ is reciprocal; that is, $\mathrm{II}_K(t^{-1}) = \mathrm{II}(t)$.



K $\quad$ $-K$ $\quad$ K^* $\quad$ K^+

@ $W_{-K}(t) = W_K(t^{-1})$, $\quad W_{K^*}(t) = W_{K^+}(t) = -W_K(t^{-1})$.

Lem. (i) $\mathrm{I}_{-K}(t) = \mathrm{I}_{K^*}(t) = \mathrm{I}_{K^+}(t) = \mathrm{I}_K(t^{-1})$

(ii) $\mathrm{II}_{-K}(t) = \mathrm{II}_{K^*}(t) = \mathrm{II}_{K^+}(t) = \mathrm{II}_K(t)$

◎ $c(K) \geq |\operatorname{maxdeg} W_K(t)| + 1,\ |\operatorname{mindeg} W_K(t)| + 1.\quad (K \neq O)$

$\boxed{\begin{array}{l} \underline{\text{Prop.}}\ \text{(i)}\ c(K) \geq |\operatorname{maxdeg} I_K(t)| + 1,\ |\operatorname{mindeg} I_K(t)| + 1. \\ \text{(ii)}\ c(K) \geq \operatorname{maxdeg} II_K(t) + 1 \end{array}}$

◎ $W_K(1) = W_K'(1) = 0,\quad I_K(1) = II_K(1) = 0.$

$\boxed{\underline{\text{Prop.}}\quad I'(1) = II'(1) = 0.}$

◎ $J(K)$: the odd writhe of K is the sum of the coefficients of terms of odd degree of $W_K(t)$. ← always even.

$\boxed{\begin{array}{l} \underline{\text{Prop.}}\ \text{(i)}\ I_K(t) = \sum a_k t^k \Rightarrow \sum a_{2\ell+1} \equiv 0 \pmod 2. \\ \text{(ii)}\ II_K(t) = \sum b_k t^k \Rightarrow \sum b_{2\ell+1} \equiv 0 \pmod 4. \end{array}}$

◎ $f(t) \in \mathbb{Z}[t,t^{-1}]$, $\exists K$ s.t. $W_K(t)=f(t) \Leftrightarrow f(1)=f'(1)=0$.

<u>Thm.</u> For $f(t) \in \mathbb{Z}[t,t^{-1}]$, the following are equivalent.

(i) $\exists K$ s.t. $\mathrm{I}_K(t)=f(t)$.

(ii) $f(1)=f'(1)=0$.

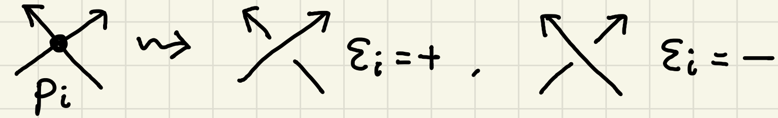
<u>Thm.</u> For $f(t) \in \mathbb{Z}[t,t^{-1}]$, the following are equivalent.

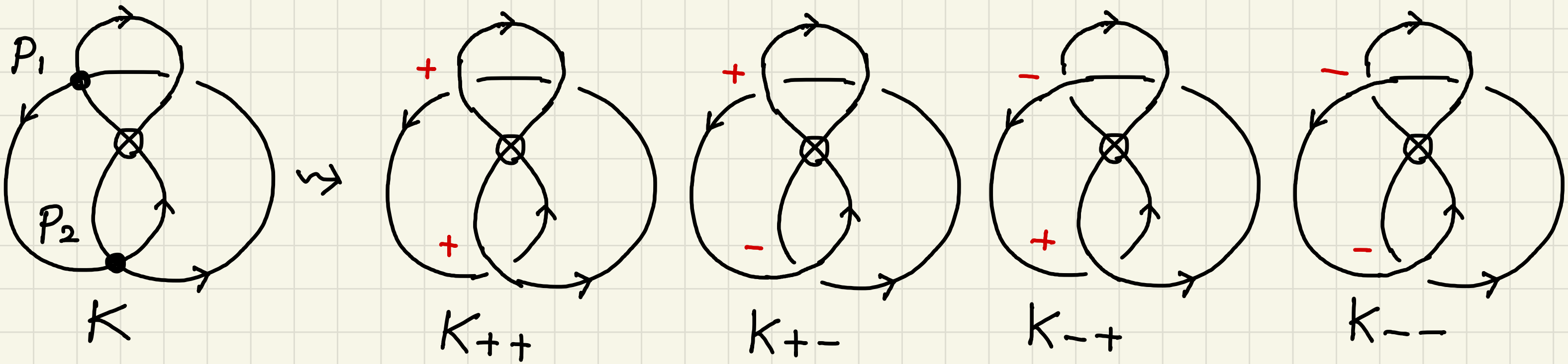
(i) $\exists K$ s.t. $\mathrm{II}_K(t)=f(t)$.

(ii) $\begin{cases} f(1)=0, \\ f(t) \text{ is reciprocal}, \\ (\text{the sum of the coefficients of odd terms}) \equiv 0 \pmod 4 \end{cases}$

<u>Ex.</u> $f(t)=t-2+t^{-1}$ is realized by $\mathrm{I}_K(t)$, but not by $\mathrm{II}_K(t)$.

K : a singular virtual knot with m double points $p_1, \cdots, p_m$

$K_{\varepsilon_1 \cdots \varepsilon_m}$: the virtual knot by 



◎ $v(K)$: a finite type invariant of order m

$\Leftrightarrow$ (i) $\displaystyle\sum_{\varepsilon_i} \varepsilon_1 \cdots \varepsilon_{m+1} v(K_{\varepsilon_1 \cdots \varepsilon_{m+1}}) = 0$, $\forall K$ with $m+1$ double pts,

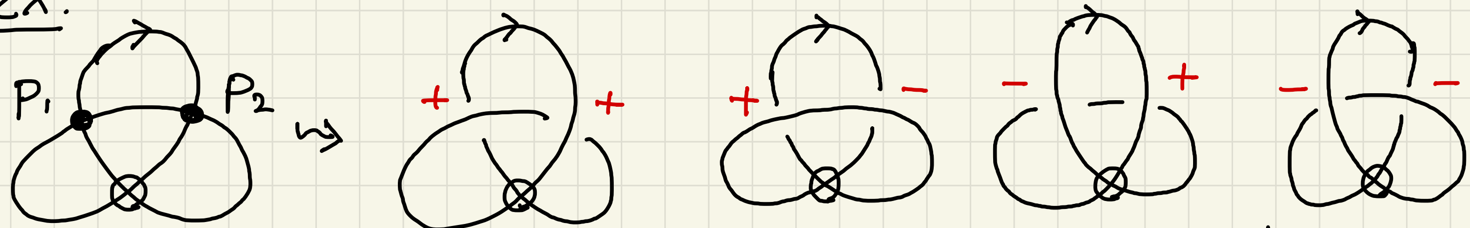
(ii) $\displaystyle\sum_{\varepsilon_i} \varepsilon_1 \cdots \varepsilon_m v(K_{\varepsilon_1 \cdots \varepsilon_m}) \neq 0$, $\exists K$ with m double pts.

◎ $W_K(t)$ is a fin. type inv. of order 1.

$$W_{K_{++}}(t) - W_{K_{+-}}(t) - W_{K_{-+}}(t) + W_{K_{--}}(t) = 0.$$

Thm. $I_K(t)$, $II_K(t)$ are fin. type inv. of order 2

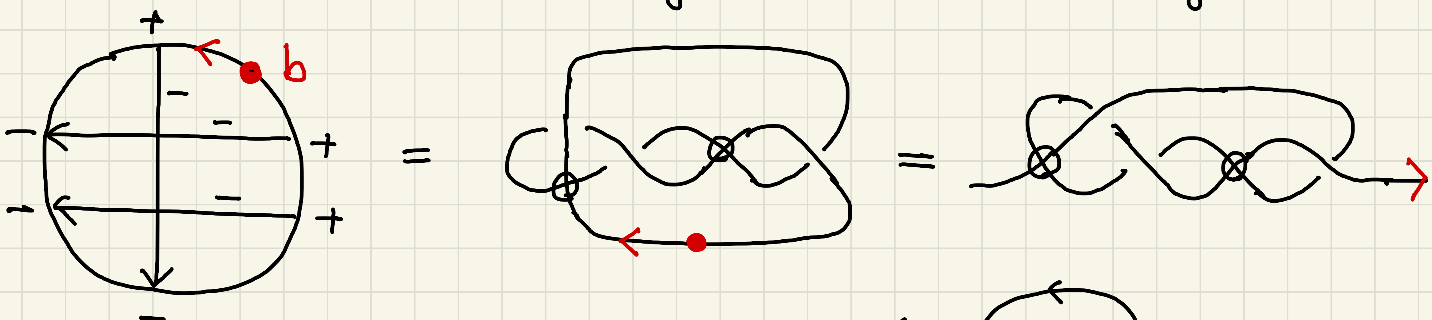
Ex.



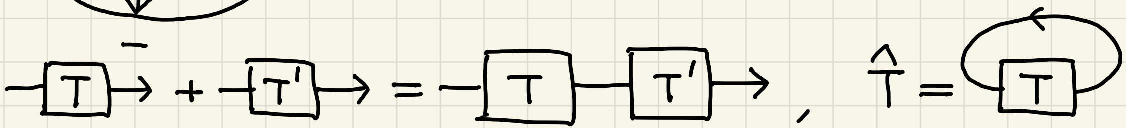
	K	$K_{++} = 2.1$	K_{+-}	K_{-+}	$K_{--} = 2.1^*$
$W_K(t)$		$t - 2 + t^{-1}$	0	0	$-t + 2 - t^{-1}$
$I_K(t)$		$-t + 2 - t^{-1}$	0	0	$-t + 2 - t^{-1}$
$II_K(t)$		$-2t + 4 - 2t^{-1}$	0	0	$-2t + 4 - 2t^{-1}$.

§5. Connected sum.

T: a dotted virtual knot (a long virtual knot, a 1-string virtual tangle)



②



(the closure of T).

Prop. $W_T^0 = \sum_{r_i \not\ni b} (t^{r_i \cdot \bar{r}_i} - 1)$, $W_T^1(t) = \sum_{r_i \ni b} (t^{r_i \cdot \bar{r}_i} - 1)$: inv. of T.

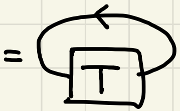
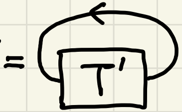
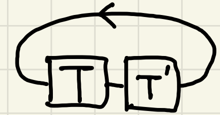
Rem. (i) $W_T^0(t) + W_T^1(t) = W_{\hat{T}}(t)$.

(ii) $W_T^0(1) = W_T^1(1) = 0$.

Ex. $W_{\hat{T}}(t) = -2t + 3 - t^{-2}$, $W_T^0(t) = -2t + 2$, $W_T^1(t) = 1 - t^{-2}$

K, K' : virtual knots

$\mathcal{C}(K,K') := \{ \widehat{T+T'} \mid \hat{T} = K,\ \hat{T}' = K' \}$

$K =$  $K' =$  

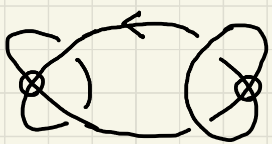
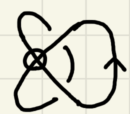
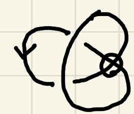
A virtual knot in $\mathcal{C}(K,K')$ is called a connected sum of K and K'.

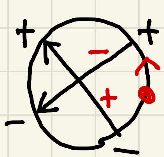
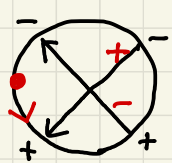
⊚ $\forall K'' \in \mathcal{C}(K,K')$, $W_{K''}(t) = W_K(t) + W_{K'}(t)$.

Thm. (i) $\mathrm{I}_{\widehat{T+T'}}(t) = \mathrm{I}_{\hat{T}}(t) + \mathrm{I}_{\hat{T}'}(t) + W_T^0(t)\,W_{T'}^1(t) + W_T^1(t)\,W_{T'}^0(t)$.

(ii) $\mathrm{II}_{\widehat{T+T'}}(t) = \mathrm{II}_{\hat{T}}(t) + \mathrm{II}_{\hat{T}'}(t) + W_T^0(t)\,W_{T'}^0(t^{-1}) + W_T^1(t)\,W_{T'}^1(t^{-1})$

$+ W_T^0(t^{-1})\,W_{T'}^0(t) + W_T^1(t^{-1})\,W_{T'}^1(t)$.

Ex. $K =$  is a conn. sum of   (triv. knots)

$T =$  , $T' =$ 

$$\begin{cases} W_T^0(t) = -t+1, & W_T^1(t) = t-1 \\ W_{T'}^0(t) = 1-t^{-1}, & W_{T'}^1(t) = -1+t^{-1} \end{cases}$$

$$\Rightarrow \begin{cases} \mathrm{I}_K(t) = 2t - 4 + 2t^{-1}, \\ \mathrm{II}_K(t) = 2t^2 - 4t + 4 - 4t^{-1} + 2t^{-2}. \end{cases}$$

Prop. $\forall K$: a virtual knot, $\forall f(t) \in \mathbb{Z}[t, t^{-1}]$ with $f(1) = 0$
$\exists T$: a dotted virtual knot
s.t. $\hat{T} = K$, $W_T^0(t) = f(t)$, and $W_T^1(t) = W_K(t) - f(t)$

Thm. $\forall K, K'$, $\mathcal{C}(K, K')$ is infinite.

☺ $\exists T_n$ s.t. $\hat{T}_n = K$, $W_{T_n}^0(t) = t^n - 1$, $W_{T_n}^1(t) = W_K(t) - (t^n - 1)$.
$\exists T_n'$ s.t. $\hat{T}_n' = K'$, $W_{T_n'}^0(t) = t^n - 1$, $W_{T_n'}^1(t) = W_{K'}(t) - (t^n - 1)$.

$$\Rightarrow \mathrm{I}_{\widehat{T_n + T_n'}}(t) = \mathrm{I}_K(t) + \mathrm{I}_{K'}(t) + \big(W_K(t) + W_{K'}(t)\big)(t^n - 1) - 2(t^n - 1)^2$$ ■

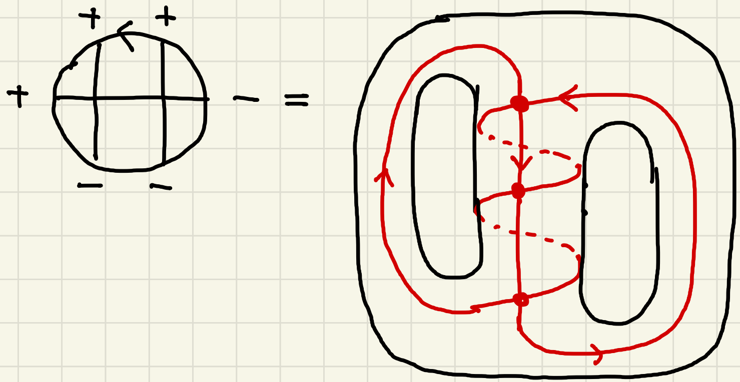
Rem. $f_{00}(D), f_{11}(D)$ $(\mathrm{mod}\ W_K(t)+W_K(t^{-1}))$ are invariants of K.

$K=4.13 \Rightarrow W_K(t)=\mathrm{I}_K(t)=\mathrm{II}_K(t)=0$ but $f_{00}(K)=2t-4+2t^{-1}$

$\Rightarrow K \neq$ classical.

Rem. $\mathrm{I}_K(t)+\mathrm{I}_K(t^{-1})-\mathrm{II}_K(t)$ are invariants under crossing changes.

$\Rightarrow$ it defines an invariant of homotopic curves on Σ.



$$\mathrm{I}_K(t)+\mathrm{I}_K(t^{-1})-\mathrm{II}_K(t)$$
$$=-6+4(t+t^{-1})-(t^2+t^{-2}).$$