Witten - Reshetikhin - Turaev function for a knot in Seifert manifolds

joint work with H. Fuji, H. Murakami, Y. Terashima

(arXiv : 2007.15872)

§1 Motivation

- Witten 1989 : Quantum field theory and the Jones polynomial

$$\begin{cases} M : \text{oriented 3-manifold} \\ L = L_1 \sqcup \cdots \sqcup L_m : \text{link in } M \\ R = (R_1, \cdots, R_m) : \text{rep's of } SU(2) \\ k \in \mathbb{Z}_{\geq 1} : \text{level} \end{cases}$$

$$\longrightarrow \quad Z_k(M,L,R) \text{ "=" } \int \underset{\vdots}{DA} \cdot W_{L,R}(A) \cdot e^{2\pi i k S_{CS}(A)}$$

integration over all SU(2)-connections / gauge

$$S_{CS}(A) = \frac{1}{8\pi^2} \cdot \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

: Chern - Simons action

$$W_{L,R}(A) = \prod_{i=1}^{m} \mathrm{Tr}_{R_i} P\exp\left(\oint_{L_i} A\right) \quad : \text{Wilson loop}$$

$$\tau_k(M,L,R) \text{ ":=" } Z_k(M,L,R) \big/ Z_k(S^3)$$

Mathematical interpretations

- Reshetikhin – Turaev 1991 ··· surgery formula
- Kohno 1992 ··· Heegaard splitting

If M is obtained from S^3 by a surgery along a link $\tilde{L} = \tilde{L}_1 \sqcup \cdots \sqcup \tilde{L}_n$

(p_1, q_1) ... (p_n, q_n) ⇝ surgery data

$$\Rightarrow \quad \tau_k(M) := (\text{factor}) \cdot \sum_{1 \le \alpha_1, \cdots, \alpha_n \le k} J_{\alpha_1 \cdots \alpha_n}(\tilde{L}, q) \cdot \prod_{j=1}^{n} d_{p_j, q_j}$$

$\tau_k(M)$: WRT invariant

$J_{\alpha_1 \cdots \alpha_n}(\tilde{L}, q)$: colored Jones polynomial for $\tilde{L}$ with $q = \exp\left(\frac{2\pi i}{k+2}\right)$

$\tau_k(M)$ is invariant under Kirby moves of $\tilde{L}$

($\Rightarrow$ topological invariant of 3-manifold)

Question

Can we upgrade $k \in \mathbb{Z}$ ⇝ $k \in \mathbb{C}$?

(q: root of 1 ⇝ q: general ?)

Witten 1989 also proposed an asymptotic formula:

$$Z_k(M) \underset{k \to +\infty}{\sim} \sum_{\alpha} e^{2\pi i (k+2) S_{CS}(\alpha)} \cdot k^{\bullet} \cdot \sum_{n=0}^{\infty} a_{\alpha,n} \cdot k^{-n}$$

α: flat $SU(2)$ connections

$\updownarrow$

critical points of S_{CS}

leading term

$\updownarrow$

Reidemeister torsion (Ray-Singer)

(Analogy to oscillatory integral (saddle point expansion)

$$\int_{\Gamma_\alpha} \exp\left(\frac{1}{\hbar} F(x)\right) dx \underset{\hbar \to 0}{\sim} \frac{e^{\frac{1}{\hbar} F(x_\alpha)}}{\sqrt{H_F(x_\alpha)}} \cdot \hbar^{\bullet} \cdot \sum_{n=0}^{\infty} a_{\alpha,n} \hbar^n$$

Γ_α: Lefschetz thimble attached to a critical point x_α ($dF(x_\alpha) = 0$))

c.f., Witten 2010:

Analytic continuation of Chern-Simons theory

$\rightsquigarrow$ Potential application to

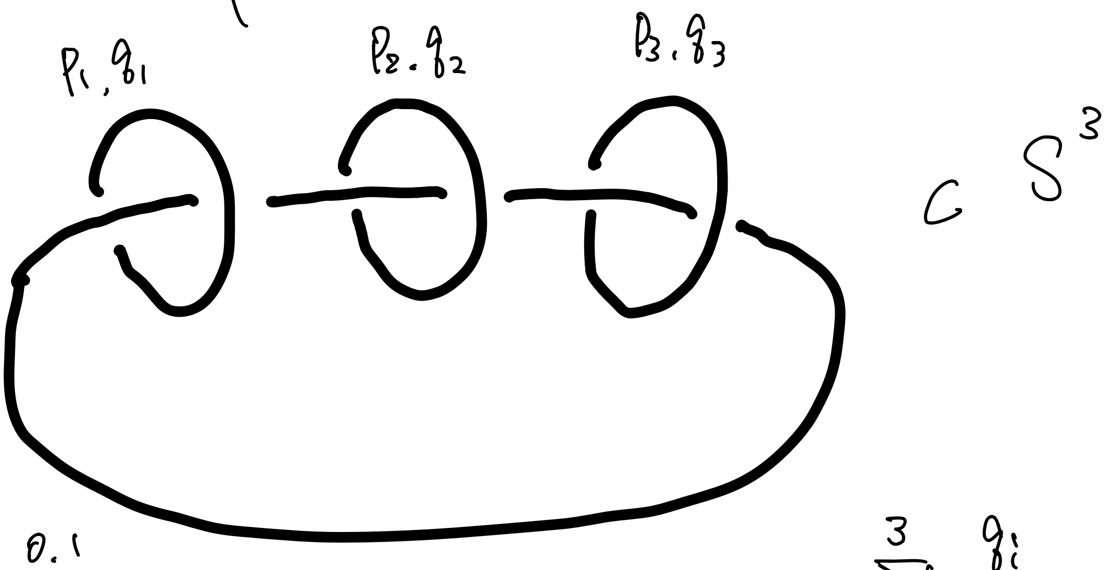
(generalized) volume conjecture ?

Suggestive result by Lawrence - Zagier 1999

$M = \Sigma(p_1, p_2, p_3)$: Brieskorn homology sphere

p_1, p_2, p_3 : pairwise coprime

surgery



$$p_1 p_2 p_3 \cdot \sum_{i=1}^{3} \frac{q_i}{p_i} = 1$$

$$\tau_K(\Sigma(p_1,p_2,p_3)) = (\text{factor})$$

$K = k+2$

$$\times \sum_{\substack{\alpha=0 \\ K \nmid \alpha}}^{2PK-1} e^{-\frac{\pi i}{2KP}\alpha^2} \cdot \frac{\prod_{j=1}^{3}\left(e^{\frac{\pi i \alpha}{K p_j}} - e^{-\frac{\pi i \alpha}{K p_j}}\right)}{e^{\frac{\pi i \alpha}{K}} - e^{-\frac{\pi i \alpha}{K}}}$$

$P = p_1 \cdot p_2 \cdot p_3$

[Lawrence - Rozansky 1999]

For $\Sigma(2,3,5)$: Poincaré homology sphere,

Lawrence – Zagier introduced a q-series

$$A(q) := \sum_{n=1}^{\infty} \underbrace{\chi_{+}(n)}_{\text{periodic function with values in } \{-1,0,+1\}} \cdot q^{\frac{n^2-1}{120}}$$

$\leftarrow$ Eichler integral of a modular form

$n \pmod{60}$	1	11	19	29	31	41	49	59	others
$\chi_{+}(n)$	1	1	1	1	-1	-1	-1	-1	0

$$A(q) = 1 + q + q^3 + q^7 - q^8 - q^{14} - q^{20} - \cdots$$

$A(q) \in \mathbb{Z}\{q^{+}\}$: convergent on $|q| < 1$

Theorem [LZ 1999]

$$\lim_{t \to 0^+} \left(1 - \frac{1}{2} A\left(q = e^{\frac{2\pi i}{K}} \cdot e^{-t}\right)\right)$$

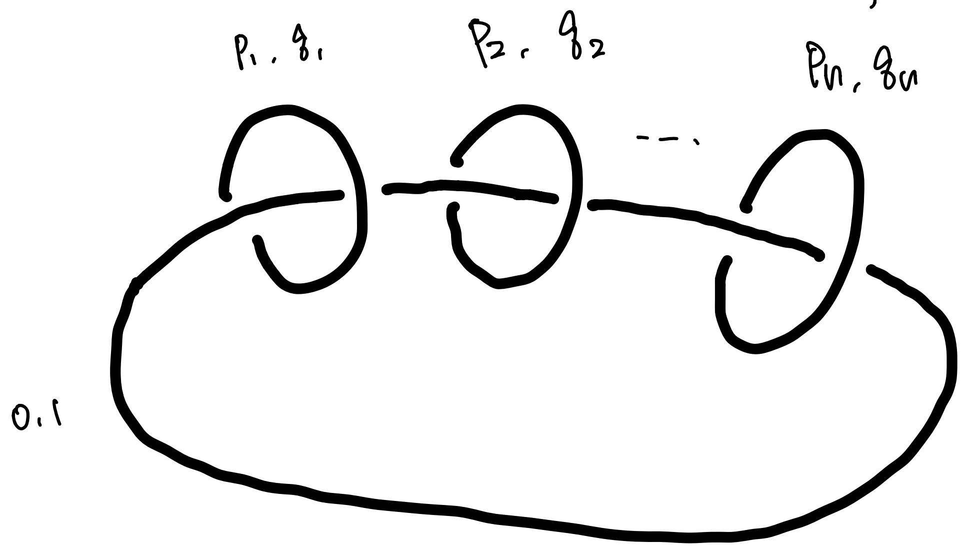
$$= (\cdots) \times \sum_{\substack{\alpha=0 \\ K \nmid \alpha}}^{2PK-1} e^{-\frac{\pi i}{2KP}\alpha^2} \cdot \frac{\prod_{j=1}^{3}\left(e^{\frac{\pi i \alpha}{K P_j}} - e^{-\frac{\pi i \alpha}{K P_j}}\right)}{e^{\frac{\pi i \alpha}{K}} - e^{-\frac{\pi i \alpha}{K}}}$$

with $(P_1, P_2, P_3) = (2,3,5)$

- Generalization by Hikami 2004, 2006, 2011

to Seifert homology sphere $\Sigma(P_1, \dots, P_n)$

$\uparrow$ surgery



$$P_1 \cdots P_n \cdot \sum_{j=1}^{n} \frac{q_j}{P_j} = 1$$

§2. Results

colored by N-dim rep

Consider a Seifert loop $X = (\Sigma(P_1, \dots, P_n), L)$

$$\rightsquigarrow \quad T_X(K;N) = (\text{factor})$$

$$\times \sum_{\substack{\alpha=0 \\ K \nmid \alpha}}^{2PK-1} e^{-\frac{\pi i}{2KP}\alpha^2} \cdot \frac{e^{N\frac{\pi i \alpha}{K}} - e^{-N\frac{\pi i \alpha}{K}}}{e^{\frac{\pi i \alpha}{K}} - e^{-\frac{\pi i \alpha}{K}}} \cdot \frac{\prod_{j=1}^{n} \left(e^{\frac{\pi i \alpha}{K \cdot P_j}} - e^{-\frac{\pi i \alpha}{K \cdot P_j}} \right)}{\left(e^{\frac{\pi i \alpha}{K}} - e^{-\frac{\pi i \alpha}{K}} \right)^{n-2}}$$

(Previous case : $n=3$ & $N=1$)

Let us introduce WRT function for X as follows :

$$\Phi_X(q;N) := (\text{factor})$$

$$\times \sum_{\ell=-\frac{N-1}{2}}^{\frac{N-1}{2}} \cdot \sum_{\substack{(\varepsilon_1,\cdots,\varepsilon_n) \\ \in \{\pm 1\}^n}} \cdot \sum_{m=0}^{\infty} \varepsilon_1 \cdots \varepsilon_n \binom{m+n-3}{n-3} \cdot q^{\frac{P}{4}\left(2m+2\ell+n-2+\sum_{j=1}^{n}\frac{\varepsilon_j}{p_j}\right)^2}$$

$$\Uparrow$$

$$\underbrace{(\text{factor})}_{\text{depens also on } q} \cdot q^{\min} \cdot \mathbb{Z}\{q\}$$

e.g.,

- $n=2 \Rightarrow \Phi_X(q;N) = \dfrac{q^{\frac{N}{2}}-q^{-\frac{N}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} \cdot J_{T_{p_1 p_2}}(q;N)$

where $J_{T_{p_1,p_2}}(q;N) = \dfrac{q^{\frac{p_1p_2}{4}(1-N^2)}}{q^{\frac{N}{2}}-q^{-\frac{N}{2}}} \times \displaystyle\sum_{\ell=-\frac{N-1}{2}}^{\frac{N-1}{2}} \left(q^{p_1p_2\ell^2-(p_1+p_2)\ell+\frac{1}{2}} - q^{p_1p_2\ell^2-(p_1-p_2)\ell-\frac{1}{2}} \right)$

$=$ colored Jones polynomial for (p_1,p_2)-torus knot

[Morton 1995, $\cdots$] (with the normalization $J_{\text{unknot}}(q;N)=1$)

- $n=3$, $N=1$, $(p_1,p_2,p_3) = (2,3,5)$

$$\Rightarrow \Phi_X(q;N=1) = (\text{factor}) \times \left(-q^{\frac{1}{120}}\right) \times \left(1-q-q^3-q^7+q^8+q^{14}+q^{20}+\cdots\right)$$

$$\Updownarrow \text{ agree !}$$

$$1-\frac{1}{2}A_{LZ}(q) = \frac{1}{2}\left(1-q-q^3-q^7+q^8+q^{14}+q^{20}+\cdots\right)$$

Theorem [Fuji - I - Murakami - Terashima 2020]

(1) For any $k \in \mathbb{Z}_{\geq 1}$, we have

WRT invariant

$$\lim_{t \to 0+} \Phi_X\left(q = e^{\frac{2\pi i}{k}} \cdot e^{-t} ; N\right) = \tau_X(k ; N)$$

(2) The WRT function is obtained as

the "median sum" (= a version of Borel sum)

of the perturbative part (contribution of trivial conn.)

$$Z_X^{\text{pert}}(k ; N) = \sum_{n=0}^{\infty} a_n \cdot k^{-n-\frac{1}{2}}$$

(c.f., [Gukov - Mariño - Putrov 2017])

(3) $\exists \hat{A}(\hat{m}, \hat{\ell}, q)$: q-difference operator

$$\begin{cases} \hat{m}\, \Phi(q ; N) = q^{\frac{N}{2}} \Phi(q ; N) \\ \hat{\ell}\, \Phi(q ; N) = \Phi(q ; N+1) \end{cases} \qquad \hat{\ell}\hat{m} = q^{\frac{1}{2}} \hat{m}\hat{\ell}$$

s.t,

$$\hat{A}(\hat{m}, \hat{\ell}, q) \cdot \Phi_X(q ; N) = 0$$

and its classical limit is a component of the zero locus of the A-polynomial of $\Sigma(p_1 \cdots p_n) / \mathrm{Tub}(L)$

Classical limit :

$$A(m,\ell) = (\ell-1)(\ell-m^{-p})(\ell+m^{p}) = 0$$

$$\left(\overset{?}{\parallel} \quad A_K(m,\ell) \right)$$

§3. How Borel summation produce q-series ?

(Gevrey-1) divergent series $\xrightarrow[\text{sum}]{\text{Borel}}$ holomorphic function

$$f(k) = \sum a_n k^{-n} \qquad\qquad (\mathcal{S}f)(k)$$

s.t., $(\mathcal{S}f)(k) \sim f(k)$ when $k \to \infty$

Borel transform (= term-wise inverse Laplace transform) $\downarrow$ $\qquad\qquad$ $\uparrow$ Laplace transform

$$f_B(\xi) = \sum \frac{a_n}{(n-1)!} \xi^{n-1}$$

Application of Borel summation to Chern-Simons theory

[Witten 2010], [Costin-Garoufalidis 2011], [Gukov-Mariño-Putrov 2017], …

- GMP 2017 : "Resurgence in complex Chern-Simons theory"

perturbative expansion of $Z_k(M)$ $\xrightarrow[\text{Borel sums}]{\text{Average of}}$ q-series (with $\mathbb{Z}$-coefficients ?)

<u>Conj</u>

limit value of the q-series at $q = e^{\frac{2\pi i}{k}}$ $= Z_k(M)$

Lawrence-Rozansky's trick : $\Sigma \to \int$

<u>Lemma</u> (c.f., [Lawrence-Rozansky 1999], [Beasley 2009])

$$\tau_X(k; N) = (\text{factor})$$

$$\times \left[\int_{\mathbb{R} e^{\frac{\pi i}{4}}} e^{k g(y)} \cdot F(y)\, dy - 2\pi i \cdot \sum_{m=1}^{2P-1} \operatorname*{Res}_{y = 2\pi i m} \frac{e^{k g(y)} \cdot F(y)}{1 - e^{-ky}}\, dy \right]$$

where

$$g(y) = \frac{i y^2}{8\pi P}, \quad F(y) = \frac{e^{\frac{Ny}{2}} - e^{-\frac{Ny}{2}}}{e^{\frac{y}{2}} - e^{-\frac{y}{2}}} \cdot \frac{\prod_{j=1}^{n} \left(e^{\frac{y}{2P_j}} - e^{-\frac{y}{2P_j}} \right)}{\left(e^{\frac{y}{2}} - e^{-\frac{y}{2}} \right)^{n-2}}$$

($n=2$: Kashaev - Tirkkonen (integral expression of Jones poly for torus knot))

$$Z^{\mathrm{triv}}(k) := \int_{\mathbb{R}e^{\frac{\pi i}{4}}} e^{k g(y)} \cdot F(y)\, dy$$: contribution from triv. conn.

$$\underset{k \to +\infty}{\sim} \sum_{n=0}^{\infty} a_n \cdot k^{-n-\frac{1}{2}} =: Z^{\mathrm{pert}}(k)$$

$$\in k^{-\frac{1}{2}} \mathbb{C}[[k^{-1}]]$$

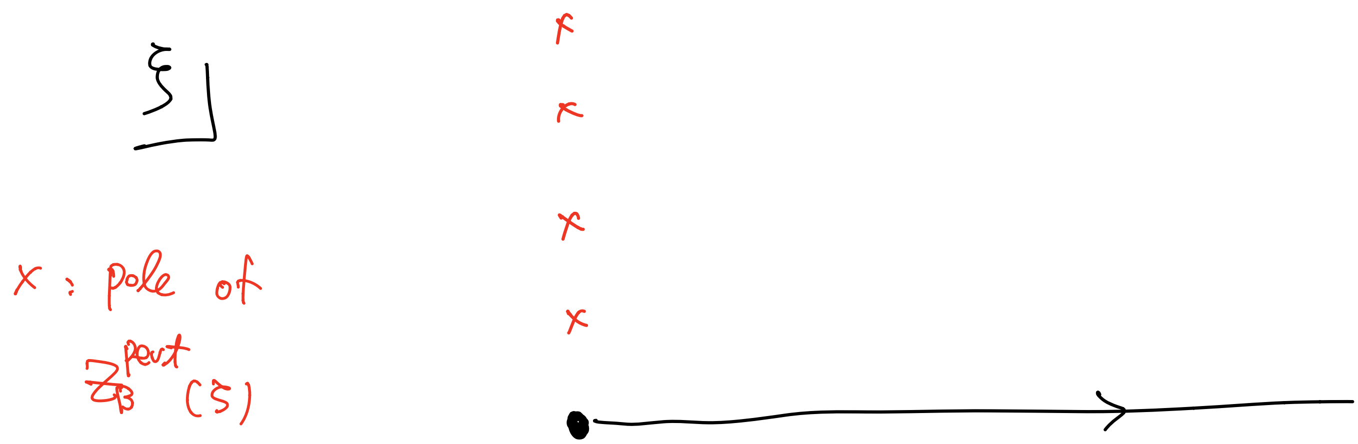
Now we can upgrade $k \in \mathbb{Z} \rightsquigarrow k \in \mathbb{C}$

Borel transform

$$Z_B^{\mathrm{pert}}(\xi) := \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+\frac{1}{2})} \xi^{n-\frac{1}{2}}$$: convergent

Lem

$$Z_B^{\mathrm{pert}}(\xi) = \left[\frac{4\pi i P}{y} \cdot F(y)\right]_{y=\sqrt{8\pi i P \xi}} - \left[\frac{4\pi i P}{y} \cdot F(y)\right]_{y=-\sqrt{8\pi i P \xi}}$$



$$(\mathcal{S}Z^{pert})(k) := \int_0^\infty e^{-k\xi} Z_B^{pert}(\xi)\, d\xi$$: Borel sum of $Z^{pert}(k)$

$Z^{pert}(k)$ is not Borel summable in the direction $\frac{\pi}{2}$ due to Borel singularities.

Thm

The following relation holds when $\operatorname{Im} k < 0$:

$$\Phi_X(q = e^{\frac{2\pi i}{k}} ; N)$$

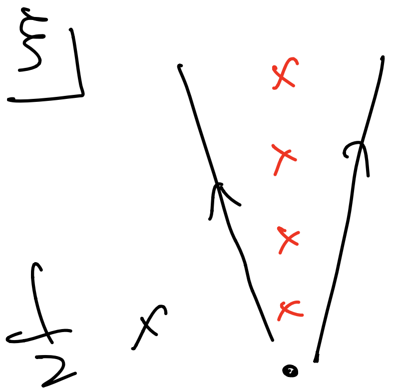
$$= (\text{factor}) \cdot \frac{\mathcal{S}_{\frac{\pi}{2}-\delta} Z^{pert}(k) + \mathcal{S}_{\frac{\pi}{2}+\delta} Z^{pert}(k)}{2}$$

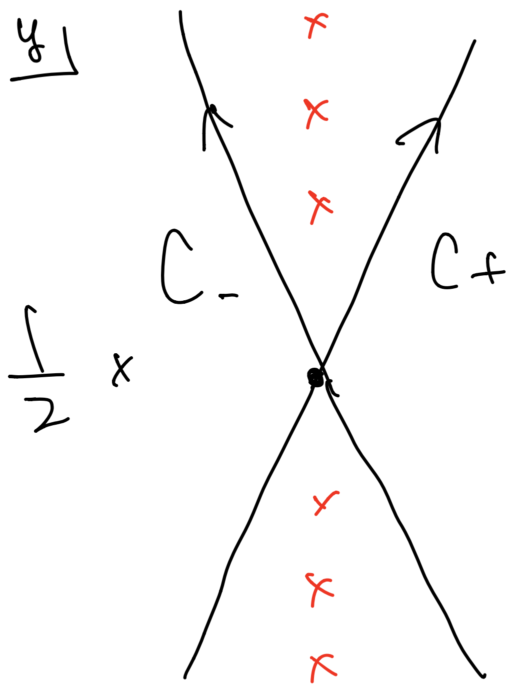
coincides with "median sum"

(proof)

R.H.S

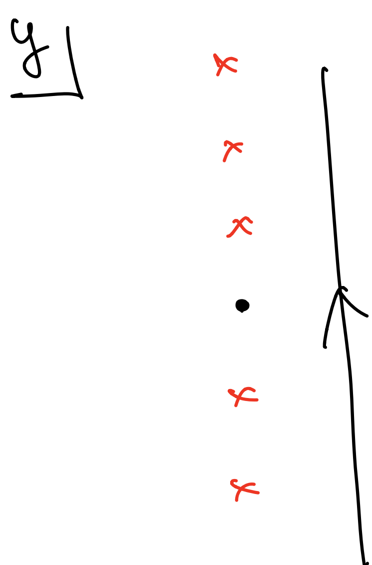
$$= \frac{1}{2}\cdot\left(\int_{C_+} + \int_{C_-}\right) e^{\kappa g(y)} F(y)\, dy$$

$\frac{1}{2}\times$ 

$=$ $\frac{1}{2}\times$ 

$$y^2 = 8\pi i P\xi$$

$$\left(e^{\kappa g(y)} = e^{-\kappa\xi}\right)$$

$=$ 

$$e^{k g(y)} F(y)$$

$$= e^{\frac{ik}{8\pi p} y^2} \cdot \frac{e^{\frac{Ny}{2}} - e^{-\frac{Ny}{2}}}{e^{\frac{y}{2}} - e^{-\frac{y}{2}}} \cdot \frac{\prod_{j=1}^{n} \left(e^{\frac{y}{2p_j}} - e^{-\frac{y}{2p_j}} \right)}{\left(e^{\frac{y}{2}} - e^{-\frac{y}{2}} \right)^{n-2}}$$

$|e^{-y}| < 1$ on the integration contour $\longrightarrow \|$

$$\sum_{l=-\frac{N-1}{2}}^{\frac{N-1}{2}} e^{-ly} \cdot \sum_{(\varepsilon_1 \cdots \varepsilon_n) \in \{\pm 1\}^n} \varepsilon_1 \cdots \varepsilon_n \, e^{\left(\sum_{j=1}^{n} \frac{\varepsilon_j}{p_j} \right) \frac{y}{2}}$$

$$\times \; e^{(n-2)\frac{y}{2}} \sum_{m=0}^{\infty} \binom{-(n-2)}{m} \left(-e^{-y} \right)^m$$

$$\shortparallel$$

$$\binom{m+n-3}{n-3}$$

Term-wise Gaussian integral:

$$\int_{\varepsilon+i\mathbb{R}} e^{Ay^2+By}\, dy$$

$$= (\cdots) \cdot e^{-\frac{B^2}{4A}}$$

$$\downarrow \quad A = \frac{i\tau}{8\pi p}$$

$$= (\cdots)\, e^{-\frac{B^2}{\frac{i\tau}{2\pi p}}}$$

$$= (\cdots) \left(e^{\frac{2\pi i}{\tau}}\right)^{\frac{B^2}{4p}}$$

$\rightarrow$ We get WRT function!

Problems

- Non homology sphere Seifert loops and more general 3-manifolds ?
- Volume Conjecture
- AJ conjecture
- Topological recursion
- What is counted by the coefficients ?

⋮

Thank you for your attention !