Mirror Symmetry: Theme and Variations
Lecture I

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Theme: Mirror Symmetry and its Origins in String Theory

- **Polchinski, String Theory (two volumes)**
- **Morrison, hep-th/9512016**

My theme is mirror symmetry and its origins in string theory. In non-perturbative string theory, mirror symmetry is a proposed equivalence between the compactification of two of the string theories (type IIA and type IIB) on different Calabi-Yau threefolds, called a mirror pair. This equivalence implies that the open-string sectors of these theories (which only exist non-perturbatively) are also equivalent: i.e., there is an equivalence of D-brane state spaces.

Mirror symmetry

Loosely speaking, *mirror symmetry for Calabi–Yau threefolds* is the assertion that Calabi–Yau threefolds come in mirror pairs (X, Y), with isomorphisms between their parameter spaces which reverse the rôles of algebro-geometric moduli and complexified Kähler cones. More generally, the mirror relationship is supposed to induce isomorphisms between various physical theories associated to the pair:

- string theories of types IIA and IIB compactified on the two members of the pair, and
- 2D conformal field theories with "$\mathcal{N} = (2, 2)$ supersymmetry",
- the spectra of BPS D-branes on those string theories.

(This last one is closely related to Kontsevich's Homological Mirror Symmetry conjecture.)

Mirror symmetry

The loose statement of mirror symmetry which we gave on the previous slide is incorrect: the complexified Kähler cone of Y can at best give a small neighborhood within the moduli space of X.

It fact, it was realized within the first few years of studying mirror symmetry that accurately identifying a mirror pair involved finding an appropriate boundary point in the compatified moduli space of X, a neighborhood of which would be the complexified Kähler cone of Y.

To understand the statements of these various physical equivalences, we need at least a rudimentary understanding of the terminology.

Non-perturbative string theory

- Relativistic classical physical theories are described in terms of fields propagating in spacetime, transofrming in representations of the Poincaré algebra. Restricting to the Lorentz algebra, fields whose associated representation does not lift from the Lorentz group to its double cover (the indefinte Spin group) are known as fermions, and have markedly different physical properties than the bosons which transform in more familiar representations of the indefinite orthogonal group. Bosons include: scalar fields (whose spacetime representation is trivial) such as the "Higgs field," gauge fields transforming as 1-forms, gravitational fields transforming as symmetric 2-tensors, and p-form fields transforming in various antisymmetric powers of the standard representation. The physics of each of these types of field is well understood.

Non-perturbative string theory

- Supersymmetric theories are invariant under a $\mathbb{Z}/2\mathbb{Z}$-graded extension of the Poincaré algebra known as a supersymmetry algebra. An odd transformation in a supersymmetry algebra relates bosons and fermions, so that irreducible representations of the supersymmetry algebra necessarily involve both kinds of field. The possible supersymmetry algebras were classfied by Nahm in 1978 under the assumption that there is some representation which only contains fields of standard types (i.e., those known to have a physical interpretation). Without gravity, supersymmetric theories can exist in dimensions ≤ 6 while the theories with gravitational fields, known as supergravity theories, exist in dimensions ≤ 11. Most relevant to our discussion today will be the ten-dimensional supergravity theories.

Non-perturbative string theory

- A (super)string theory is a ten-dimensional quantum theory of gravity which is semi-classically approximated by one of the ten-dimensional supergravity theories equipped with quantization conditions for the p-form fields.
- It is called a "string theory" because the so-called NS-NS 2-form field has an associated string, the physics of which encodes much of the information about the string theory into a two-dimensional physical theory.
- It is called a "superstring" because the two-dimensional theory also has supersymmetry. In addition, it has conformal symmetry.
- There are five string theories, but only two are relevant to our story: type IIA and type IIB.

Non-perturbative string theory

- In type IIA string theory, the bosonic fields are a scalar field known as the dilaton, the gravitational field, the NS-NS 2-form, and the "R-R" 1-form and 3-form.
- In type IIB string theory, the bosonic fields are the dilaton, the gravitational field, the NS-NS 2-form, and the "R-R" 0-form, 2-form, and "self-dual" 4-form.
- Non-perturbatively, the type IIA and type IIB string theories include "D-branes," which are submanifolds on which open strings can end, and which support a vector bundle from which the open strings acquire a gauge field at their endpoints. (Perturbatively, one doesn't notice the open strings in these theories, only the closed strings.)
- To compactify a string theory means to study it on a spacetime of the form $X^{10-d} \times M^{1,d-1}$, where X is a compact Riemannian manifold, and to try to express the resulting theory in terms of physics on $M^{1,d-1}$ alone.

Non-perturbative string theory

- Preserving supersymmetry on the compactified theory requires that X have a covariantly-constant spinor field. Preserving supersymmetry on D-branes requires that their supporting submanifolds are calibrated cycles.
- The parameters in a compactified theory are the values of scalar fields in $M^{1,d-1}$. In the type IIA and type IIB cases, these are the dilaton, a (Ricci-flat) metric on X, a (harmonic) 2-form $B \in H^2(X, \mathbb{R}/\mathbb{Z})$, and harmonic representatives of the other p-form fields.

Calabi–Yau moduli space

A *Calabi–Yau threefold* is a Ricci-flat Kähler manifold $(X, g_{\imath\bar{\jmath}})$ with Riemannian holonomy SU(3); physicists typically equip Calabi–Yau threefolds with an extra differential form $B \in H^2(X, \mathbb{R}/\mathbb{Z})$. Since the holonomy is SU(3), there are no holomorphic 2-forms, so that $H^2(X, \mathbb{C})$ is purely of type $(1, 1)$; this implies that Calabi–Yau threefolds are algebraic varieties, i.e., algebraic threefolds with trivial canonical bundle.) Moreover, the theorems of Calabi and Yau tell us that for any algebraic threefold $\mathcal{X}$ with trivial canonical bundle, and any Kähler class there is a Ricci-flat Kähler metric in that class. It follows that the natural parameter space for Calabi–Yau threefolds (of fixed topological type, say) is the bundle of Kähler cones over the familiar algebraic geometer's moduli space.

Calabi–Yau moduli space

Including the "*B*-field" gives a bundle of complexified Kähler cones over the moduli space, which is sometimes called the *semiclassical moduli space* of the Calabi–Yau threefolds. (There are a few other technicalities in constructing this space: one should mod out by the action of the automorphism group of X on the complexified Kähler cone, and a conjecture which I made back in 1992 would ensure that this quotient is well-behaved.)

Calibrated cycles for boundary conditions come in two types: algebraic cycles (for type IIB) and special Lagrangian cycles (for type IIA).

Type IIB String Theory

When type IIB string theory is compactified on a Calabi–Yau threefold Y, the algebro-geometric moduli space of Y becomes part of a larger *semi-classical string theory moduli space of Y*, which includes not only the choice of complex structure (the algebro-geometric modulus) but also a choice of complexified Kähler form $\omega + iB \in H^2(Y, \mathbb{C})/H^2(Y, \mathbb{Z})$ and a choice of what are called *Ramond–Ramond fields*, which can be regarded as taking values in the *dual of the total Deligne cohomology group* $\bigoplus \mathbb{H}^{2n}(Y, \mathbb{Z}(n))$. More precisely, the Ramond–Ramond fields involve a cycle class in K-theory rather than the integer cohomology class which was used to defined Deligne cohomology.

Note: the adjective "semi-classical" indicates that a complete understanding of this moduli space in string theory would involve "quantum corrections" to the given description, but not all of those quantum corrections are known in detail.

Type IIB String Theory

One of the features of compactified type IIB string theory is the rôle of "BPS D-branes", which are physical objects that span an effective algebraic cycle $Z \subset Y$ as well as the noncompact four-dimensional spacetime of the physical theory, and also require the specification of a holomorphic vector bundle supported on Z. (More generally, one can consider coherent sheaves on Y with arbitrary support.) The *D-brane charge* is the corresponding element of $\bigoplus H^{n,n}_{\mathbb{Z}}(Y)$ (or its K-theory variant).

Type IIB String Theory

Under mirror symmetry, the type IIB string theory compactified on Y is supposed to produce the same physics as the type IIA string theory compactified on the mirror partner X. There can also be D-branes in the type IIA string theory, but in this case they are represented by *special Lagrangian cycles* $L \subset X$ (again equipped with a bundle) rather than by algebraic cycles.

The mathematical technology currently available to study special Lagrangian cycles is much more limited than that available to study algebraic cycles. However, there is one construction, due to Bryant, which produces a special Lagrangian cycles on a large number of Calabi–Yau threefolds X: if the threefold X is defined over $\mathbb{R}$, then any connected component of its locus of real points gives a special Lagrangian cycle L.

Variation 1: Mirror Symmetry in Perturbative String Theory

- **Green, Schwarz, Witten, Superstring theory (two volumes)**
- **Candelas, de la Ossa, Green, Parkes, Nucl. Phys. B 359 (1991) 21-74**
- **Morrison, Walcher, arXiv:0709.4028**

Variation one (perturbative mirrror symmetry): in perturbative string theory, we find an equivalence between two-dimensional conformal field theories, and boundary conditions must also coincide.

Perturbative string theory

The original approach to string theory was in terms of the physics on the so-called “worldsheet” of the string, studying this theory perturbatively in terms of a parameter known as α'. The terms in the perturbative expansion correspond to evaluting the theory on worldsheets of various genus.
The perturbative expansion is completely insensitive to the RR fields in spacetime, so those are ignored: the key (bosonic) ingredients for perturbative string theory are the dilaton, the gravitational field, and the B-field.

For theories involving open strings as well as closed strings, it should be possible to specify boundary conditions for these theories, which say what happens at the endpoints of the string.

Nonlinear sigma models

If X is a space on which a string theory is compactified, the two-dimensional physical theory living on the string can be well-approximated by a "nonlinear sigma model," which is a physical theory on the worldsheet Σ of the string whose scalar fields are maps from Σ to X. Such a theory is specified by an action or a Lagrangian, and the key ingredients of the action are a dilaton, a metric and a B-field on X.
The two-dimensional theories describing string theories should be conformally invariant, and (in the genus-one approximation) this implies that the dilaton is constant, the metric is Ricci-flat and the B-field is harmonic.

(Boundary conditions, in principle, involve calibrated cycles on the target space and vector bundles supported on them.)

Nonlinear sigma models

Quantities in such a theory can be expressed as a series with a classical contribution (involving maps $\Sigma \to X$ whose image is contractible) and quantum contributions, which are nontrivial maps $\Sigma \to X$ with Σ of fixed genus. Even when the genus is fixed, there is a series to sum: one should sum over the possible homology classes of the image.

As you may know from an earlier study of physics, the physics is dominated by maps of least action. In the current context, those maps turn out to be holomorphic maps. (More generally, if we only specify a symplectic structure on X we find pseudo-holomorphic maps.)

Nonlinear sigma models

The theories associated to Calabi–Yau threefolds have a lot of structure, stemming from their supersymmetry. There are "operators" in these theories associated to both the first-order deformations of complex structure, and to complexified Kähler classes (which can be thought of as first-order deformations of complexified Kähler structure). In addition, there are "correlation functions" associated to sets of such operators.

When evaluted on worldsheets of genus zero, the correlation function of three complex structure operators has no contributions from holomorphic maps: they are given by their "classical value". However, the correlation functions of three complexified Kähler operators have an interesting expansion receiving contributions from holomorphic rational curves of each degree.

Equating these two leads to curve-counting predictions!

The Quintic and its Mirror

- quintic: $X = \{F_5(x_1, \ldots, x_5) = 0\} \subset \mathbb{CP}^4$

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- quintic-mirror:
  $Y \to \overline{Y} = \{\frac{1}{5}\sum x_j^5 - \psi \prod x_j = 0\}/(\mathbb{Z}_5)^3 \subset \mathbb{CP}^4/(\mathbb{Z}_5)^3$
  Known from Greene–Plesser analysis of representations of the supersymmetry algebra of the 2D SCFT.

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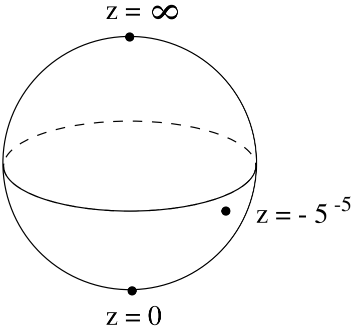
- algebro-geometric ("complex structure") moduli space, with parameter $z = (-5\psi)^{-5}$
- the identification between the two is made with the help of periods $\Phi(z) = \int_\Gamma \Omega_z$, for $\Gamma \in H_3(Y, \mathbb{Z})$ and Ω_z a holomorphic 3-form on Y_z

The Quintic and its Mirror

Explicitly,

$$\Omega_z = \operatorname{Res}\left(\frac{\sum_j (-1)^{j-1} x_j \, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_5}{\frac{1}{5}\sum x_j^5 - \psi \prod x_j}\right)$$



Periods and the Mirror Map

By differentiating under the integral sign, we can see that $\Phi(z)$ satisfies an algebraic differential equation $\mathcal{D}\Phi = 0$, where, for an appropriate choice of Ω_z,

$$\mathcal{D} = \left(z\frac{d}{dz}\right)^4 - 5z\left(5z\frac{d}{dz}+1\right)\left(5z\frac{d}{dz}+2\right)\left(5z\frac{d}{dz}+3\right)\left(5z\frac{d}{dz}+4\right)$$

It is easy to find a single power series solution near $z = 0$:

$$\Phi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5}\, z^n$$

but the other three solutions are elusive.

Periods and the Mirror Map

The recursion relations implied by the equation lead one to a formal power series of the form

$$\Phi(z,\alpha) = \sum_{n=0}^{\infty} \frac{(5\alpha+1)(5\alpha+2)\cdots(5\alpha+5n)}{[(\alpha+1)(\alpha+2)\cdots(\alpha+n)]^5}\, z^{\alpha+n}\,;$$

one finds that $\mathcal{D}(\Phi(z,\alpha)) = \alpha^4 z^\alpha$ and so we must have $\alpha^4 = 0$ in order to obtain a solution. In fact, the formal solution can be interpreted with α taken from the ring $\mathbb{C}[\alpha]/(\alpha^4)$ as follows: each coefficient

$$\frac{(5\alpha+1)(5\alpha+2)\cdots(5\alpha+5n)}{[(\alpha+1)(\alpha+2)\cdots(\alpha+n)]^5}$$

can be evaluated in that ring, and written as a polynomial in α of degree 3; moreover, z^α can be expanded as $1+\alpha\ln z+\frac{1}{2}\alpha^2(\ln z)^2+\frac{1}{6}\alpha^3(\ln z)^3$.

Periods and the Mirror Map

The coefficients of 1, α, α^2, and α^3 in the resulting expression give four linearly independent multi-valued solutions $\Phi_0(z)$, $\Phi_1(z)$, $\Phi_2(z)$, and $\Phi_3(z)$ with $\Phi_j(z)$ containing $(\ln z)^j$ and lower powers. The *mirror map* is the identification of the complexified Kähler moduli space of X with the complex moduli space of Y via

$$t = \frac{1}{2\pi i} \frac{\Phi_1(z)}{\Phi_0(z)},$$

or

$$q = \exp(\Phi_1(z)/\Phi_0(z)).$$

Three-point Functions

A key aspect of the physics is captured by the so-called *topological correlation functions*, among which is the "three-point function," a trilinear map on $H^{1,1}(X,\mathbb{C})$, resp. $H^1(Y, T_Y^{(1,0)})$. On the quintic X, given $A, B, C \in H^{1,1}(X,\mathbb{C})$, the three-point function has an expansion of the form

$$\langle \mathcal{O}_A \mathcal{O}_B \mathcal{O}_C \rangle = A \cdot B \cdot C + \sum_{0 \neq \eta \in H_2(X,\mathbb{Z})} A(\eta) B(\eta) C(\eta) N_\eta \frac{z^\eta}{1 - z^\eta},$$

where N_η counts the number of genus zero holomorphic curves in the class η, and is closely related to the Gromov–Witten invariant of X.

Three-point Functions

On the mirror quintic Y, given $\alpha, \beta, \gamma \in H^1(Y, T_Y^{(1,0)})$, the three-point function takes the form

$$\langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \rangle = \int_Y \nabla_\alpha(\Omega \lrcorner \beta) \wedge (\Omega \lrcorner \gamma),$$

and this can be readily calculated from the periods. Comparing the two yields the predictions of Candelas, de la Ossa, Green and Parkes: the generic quintic threefold has 2875 lines, 609250 conics, 317206375 twisted cubics, and so on.

The original computation of Candelas, de la Ossa, Green, and Parkes resulted in predictions for the number N_d of rational curves of degree d on the generic quintic threefold:

d	N_d
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000
7	295091050570845659250
8	375632160937476603550000
9	503840510416985243645106250
10	704288164978454686113482249750
⋮	⋮

Disk Counting

The story so far has been about closed string theory. But in the presence of D-branes, it is now understood that open strings also play a rôle. As mentioned above, a D-brane on the quintic threefold will be a special Lagrangian submanifold L of X, and open strings are expected to end on such a submanifold. Instead of counting holomorphic curves of fixed genus, the open string theory should count open Riemann surfaces whose boundary lies on L (again, one expects, of fixed genus).

Disk Counting

The specific special Lagrangian L which we use is the set of real points of a quintic threefold defined over $\mathbb{R}$. In, fact, since there are many connected components of the moduli of such real quintic threefolds (and many things about L, including its topology, depend on the component), we specialize further to the component containing the real Fermat quintic:

$$X = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subset \mathbb{CP}^4,$$

$$L = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subset \mathbb{RP}^4.$$

This L is known to have the topology of $\mathbb{RP}^3$. Some of the holomorphic curves of genus zero on X are defined over $\mathbb{R}$; others come in complex conjugate pairs. The ones defined over $\mathbb{R}$ meet L and are divided by it into a pair of disks: it is these disks which we wish to count.

Disk Counting

Just to preview the results, we will be able to do the count only for curves of odd degree, and the answer will be: for degree 1, there are 1430 complex conjugate pairs and 15 invariant curves, leading to 30 disks; for degree 3 there are 158602805 complex conjugate pairs and 765 invariant curves, leading to 1530 disks; and so on.

Disk Counting

The background for doing open string theory requires one additional piece of data, in addition to the special Lagrangian submanifold L: it requires a $U(1)$ bundle to be specified on L, with flat connection. Since $H_1(L, \mathbb{Z}) = \mathbb{Z}_2$, there are two choices for this data; we will use L_+ and L_- to denote the special Lagrangian, equipped with such a choice. The D-brane charges of $L_\pm$, suitably defined, are the same, and the difference $L_+ - L_-$ is naturally associated to a function $\mathcal{T}(t)$ which physically represents the domain wall tension for a BPS domain wall separating vacua corresponding to L_+ and L_- boundary conditions. (The "BPS" condition means that the function is given by the same kind of pairing with Ramond–Ramond fields which we discussed in the type IIB context.)

Disk Counting

On the other hand, this tension can be computed as a sum of a semi-classical term together with disk instanton corrections, which leads to an expression of the form

$$\mathcal{T}(t) = \frac{t}{2} \pm \left(\frac{1}{4} + \frac{1}{2\pi^2} \sum_{d \text{ odd}} n_d q^{d/2} \right),$$

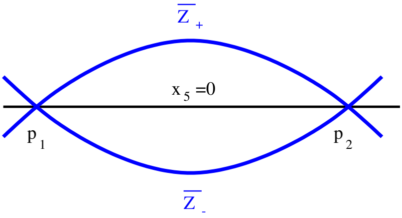
where $q = e^{2\pi i t}$ and n_d are the open Gromov–Witten invariants counting disks of degree d.

Open Mirror Symmetry

To explain the physics reasoning which leads to the identification of the mirror of $L_+ - L_-$ (an algebraic cycle on the mirror manifold Y) would take me too far afield today. Suffice it to say that Walcher and I identified this mirror cycle in the following geometric form. If we restrict the equation of $\overline{Y}$ to the plane $\mathbb{P} := \{x_1 + x_2 = x_3 + x_4 = 0\}$, we find

Open Mirror Symmetry

$$\frac{1}{5}x_5^5 - \psi x_1^2 x_2^2 x_5 = \frac{1}{5}x_5\left(x_5^2 - \sqrt{5\psi}x_1x_2\right)\left(x_5^2 + \sqrt{5\psi}x_1x_2\right).$$

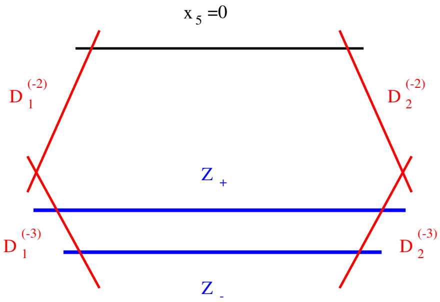


We then define

$$\overline{Z}_{\pm} = \mathbb{P} \cap \{x_5^2 \pm \sqrt{5\psi}x_1x_2 = 0\},$$

Open Mirror Symmetry

and claim that the mirror of $L_+ - L_-$ is $Z_+ - Z_-$, where $Z_\pm$ is the pullback of $\overline{Z}_\pm$ to a resolution of singularities of $\overline{Y}$.



Open Mirror Symmetry

Our remaining task is to evaluate

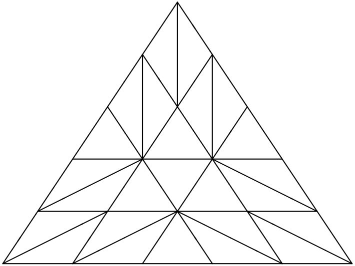
$$\widehat{\mathcal{T}}(z) := \int_\Gamma \Omega_z,$$

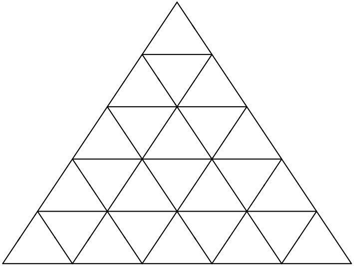
where $\partial\Gamma = Z_+ - Z_-$. The strategy is to apply the differential operator $\mathcal{D}$ to the integral. The result will not vanish, but will rather lead to an *inhomogeneous differential equation* for $\widehat{\mathcal{T}}(z)$.

(The fact that this kind of Abel–Jacobi integral will satisfy an inhomogeneous version of the differential equation satisfies by the periods themselves was observed by Griffiths in the late 1970's as he studied *normal functions.*)

Open Mirror Symmetry

The computation was surprisingly difficult, and had to be done on an explicit resolution of singularities Y of $\overline{Y}$. In particular, one had to use toric resolutions of singularities with toric data given by diagrams such as:





(This is in contrast to previous Hodge-theoretic calculations for the quintic-mirror, which could always be done on the singular space.)

Open Mirror Symmetry

The result is

$$\mathcal{D}(\widehat{\mathcal{T}}(z)) = \frac{15}{16\pi^2}\sqrt{z}.$$

The remaining steps of the computation are now straightforward: a series solution $\widehat{\mathcal{T}}(z)$ to the inhomogeneous equation can be found, and then

$$\mathcal{T}(t) = \widehat{\mathcal{T}}(z)/\Phi_0(z)$$

can be calculated and converted into a series in the mirror map parameter t. As expected, there is an ambiguity in this calculation, provided by the solutions Φ to $\mathcal{D}\Phi = 0$, but the answer is uniquely specified by insisting that the leading terms agree with the form determined by the physics:

$$\mathcal{T}(t) = \frac{t}{2} \pm \left(\frac{1}{4} + \frac{1}{2\pi^2}\sum_{d \text{ odd}} n_d q^{d/2}\right).$$

Open Mirror Symmetry

Solving for the coefficients n_d determines the numbers of disks!

The analogue of the mirror theorems of Givental and Lian–Liu–Yau in this case—the mathematical verification that these disk numbers are correct—has been carried out by Pandharipande, Solomon and Walcher. Moreover, analogues of this computation have now been made for a number of different one-parameter examples. However, multi-parameter examples have so far resisted attempts at computation, so there may well be additional aspects of this story which are yet to be understood.

Variation 2: Gauged Linear Sigma Models

- **Witten, hep-th/9301042**
- **Batyrev, alg-geom/9310003**
- **Batyrev, Borisov, alg-geom/9402002**
- **Morrison, Plesser, hep-th/9508107**
- **Jockers, Kumar, Lapan, Morrison, Romo, arXiv:1208.6244**

Variation two (GLSM): for a particular class of two-dimensional field theories ("abelian gauged linear sigma models") the equivalence is quite explicit and closely related to the combinatorial mirror symmetry discovered by Batyrev and Batyrev-Borisov.

Gauged Linear Sigma Models

Gauged linear sigma models are a class of 2D supersymmetric quantum field theories which are expected to "flow in the infrared" to conformal field theories. We illustrate the basic construction with the case of the quintic hypersurface, treating it quite mathematically at the outset. The homogeneous polynomial $f(x_1, \ldots, x_5)$ can be regarded either as describing a section of the line bundle $\mathcal{O}(-K_{\mathbb{P}^4})$, or as defining a complex-valued function

$$W(x_0, x_1, \ldots, x_5) := x_0 f(x_1, \ldots, x_5)$$

on the total space of the line bundle $\mathcal{O}(K_{\mathbb{P}^4})$ (with fiber coordinate x_0). In the latter interpretation, the Calabi–Yau threefold coincides with the critical set $\mathrm{Crit}(W)$ of the function W. In fact, if the polynomial f is transverse, the critical set away from the origin in $\mathbb{C}^5$ is defined by $x_0 = f(x_1, \ldots, x_5) = 0$.

Gauged Linear Sigma Models

(A similar construction leads from a complete intersection $f_0 = \cdots = f_{k-1} = 0$ in a projective toric variety, with f_j a section of $\mathcal{O}(L_j)$, to the polynomial function $W = x_0 f_0 + \cdots + x_{x-1} f_{k-1}$ on the total space of the bundle $\mathcal{O}(-L_0) \oplus \cdots \oplus \mathcal{O}(-L_{k-1})$. The critical set $\mathrm{Crit}(W)$ again coincides with the original complete intersection.)

The ambient space $\mathcal{O}(K_{\mathbb{P}^4})$ can itself be described by a quotient construction. We again describe it using symplectic reduction: there is an action of $U(1)$ on $\mathbb{C}^6$ defined by

$$e^{i\theta} : (x_0, x_1, \ldots, x_5) \mapsto (e^{-5i\theta} x_0, e^{i\theta} x_1, \ldots, e^{i\theta} x_5)$$

which admits a moment map $\mu : \mathbb{C}^6 \to \mathfrak{g}^* \cong \mathbb{R}^1$ given by

$$\mu(x_0, x_1, \ldots, x_5) = \frac{1}{2}\Big(-5|x_0|^2 + \sum_{j=1}^{5} |x_j|^2 \Big) ,$$

and $\mathcal{O}(K_{\mathbb{P}^4}) = \mu^{-1}(r)/G$ for appropriate values of r. Note that the polynomial function $W : \mathbb{C}^6 \to \mathbb{C}$ is G-invariant.

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The general version of this construction goes as follows: given a subgroup G of $U(1)^n$, a point $r \in \mathfrak{g}^*$ in the image of the moment map, and a G-invariant polynomial W defining a function on $\mathbb{C}^n$, we can study $\mathrm{Crit}(W)$ on the quotient variety $\mu^{-1}(r)/G$. If the polynomial W is given explicitly as

$$W(x_0, \dots, x_{n-1}) = \sum_{j=0}^{m-1} c_j \prod_{k=0}^{n-1} x_k^{p_{jk}} \qquad (*)$$

(with $c_j \neq 0$), then the combinatorics in this construction are essentially captured by the $m \times n$ matrix of exponents $P = (p_{jk})$. (We assume that W contains enough monomials so that the rank of P is $d := n - \dim G$.) The coefficients c_j are somewhat redundant: there is a group which acts on the space of polynomials of the form (*), and we must form the quotient by this group action. The true coordinates on the complex structure moduli space are the invariant quantities z_ℓ for this group action.

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The conditions which this data must satisfy are that the monomials in W generate a Gorenstein cone, and that the dual of this cone also be Gorenstein. In terms of the matrix P, this means that there must exist rational vectors μ and ν such that $P\mu = {}^t(1,\dots,1)$, and ${}^t\nu P = (1,\dots,1)$. It turns out that these conditions imply that whenever $\mathrm{Crit}(W)$ is a manifold of dimension $d - 2({}^t\nu P\mu)$, it is a Calabi–Yau manifold. Triangulations of the Gorenstein cone are needed to construct a nonsingular Calabi–Yau manifold, and these are indexed by the so-called "secondary fan."

To encode the group G in the same combinatorial structure, we introduce a basis x^{t_α} of G-invariant Laurent monomials on $\mathbb{C}^n$; then we can write

$$x^{p_j} = \prod_{\alpha=1}^{d} (x^{t_\alpha})^{s_{j\alpha}}$$

for each of the monomials occurring in W.

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This gives a factorization of P as the product of an $m \times d$ matrix S and a $d \times n$ matrix T, with the group G being completely determined by T. Changing the basis of Laurent monomials alters (S, T) to (SL^{-1}, LT) for some $L \in GL(d, \mathbb{Z})$.

The gauged linear sigma model is a physical theory built from the group G and its action on the x's. It is expected that at low energies, this theory will agree with (the perturbative part of) type II string theory compactified on the associated Calabi–Yau manifold.

There is a mirror partner of a gauged linear sigma model, whose construction is essentially due to Batyrev and Borisov. To describe the mirror partner, one merely replaces P, S, and T by their transposes. The dual group $\widehat{G}$ is determined from the ($\widehat{G}$-invariant) Laurent monomials whose exponents form the matrix ${}^tS$, and the dual polynomial $\widehat{W}$, which is a $\widehat{G}$-invariant polynomial in m variables, can be written explicitly as

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$$\widehat{W}(y_0, \ldots, y_{m-1}) = \sum_{k=0}^{n-1} \widehat{c}_k \prod_{j=0}^{m-1} y_j^{p_{jk}} \,.$$

This mirror partner is somewhat mysterious, due to the new parameters $\widehat{c}_k$ which must be introduced. However, the original group G will act on those parameters (through its action on the set of mirror polynomials), and the G-invariant quantities are familiar ones. Explicitly, if we write the moment map for the original G-action in the form $\mu(x) = \frac{1}{2}\sum_{k=1}^{n-1} \chi_k |x_k|^2$, where χ_k is the character for the action of G on the k^{th} variable, then the invariant quantities for the G action on the coefficients of $\widehat{W}$ can be described as:

$$\frac{1}{2\pi i} \sum (\log \widehat{c}_k) \chi_k \in \mathfrak{g}^*_{\mathbb{C}} / \mathfrak{g}^*_{\mathbb{Z}} \,. \qquad (**)$$

(We have written the invariants additively, introducing a logarithm, and they are thus multi-valued.)

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Mirror symmetry predicts that the imaginary part of this invariant quantity (**) is to be identified with r, i.e.,

$$r = \frac{-1}{2\pi} \sum (\log |\widehat{c}_k|) \chi_k \,.$$

(Similarly, the invariant combinations $(\log z_\ell)/2\pi i$ of the original coefficients c_j can be identified with the complexification of the Kähler parameters $\widehat{r}_\ell$ of the mirror theory.)

This construction provides a global way to identify moduli spaces, and to go beyond the small neighborhoods of large complex structure limit points. The gauged linear sigma model makes sense for arbitrary values of r, not just ones near an appropriate boundary point, and the description of a mirror theory shows that this realization could be a geometric one on the mirror partner.

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In fact, an explicit (physics) computation can be made of the locus where the theory becomes singular (aside from toric boundary points like $z = 0$ and $z = \infty$), and it reproduces the structure of the discriminant locus of the mirror polynomial, including all of its components. In the case of the quintic, there is only one component, a polynomial with a single zero, at $z = -5^{-5}$. It should be stressed that this computation is made purely from the point of view of the quintic itself, without reference to the mirror theory.

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The quantum cohomology ring also corresponds as expected from mirror symmetry. It can be precisely calculated in the gauged linear sigma model on either side (in one case from the data of the polynomial, in the other case from the toric data, refined by analyzing the physics) and the results agree. Relating this result to the enumerative predictions involves determining an appropriate basis of cohomology (or in physical terms, calculating the effect of renormalization), so one cannot derive the Mirror Theorem directly in this way; however, the proofs of the Mirror Theorem rely on similar results at some step along the way.

Applying this entire set-up to the case of the general quintic, we obtain a 6×126 matrix; the mirror partner can be determined from the transposed 126×6 matrix. However, the calculation of the geometry of the mirror would be formidable from this point of view.

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An alternative is to begin with a quintic with fewer monomials. If we start in $\mathbb{P}^4$ with the quintic defined by

$$\frac{1}{5}\sum x_j^5 - \psi \prod x_j = 0$$

then the associated factored matrix is given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

(We can choose $\mu = (1,0,0,0,0,0)$ and $\nu = (1,0,0,0,0,0)$ to obtain ${}^t\nu P \mu = 1$ and verify the conditions on the data.) The group described by this factorization is $G = U(1)$.

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To form the mirror, we take the transpose:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 5 & -1 & -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 5 & 0 \end{pmatrix}$$

This represents the same homogeneous polynomial as before; however, this time the group is $G = U(1) \times (\mathbb{Z}_5)^3$ and so the Calabi–Yau is actually a hypersurface in a quotient space $\mathbb{P}^4/(\mathbb{Z}_5)^3$.

The Two-Sphere Partition Function

There was recently a significant advance in our understand of the physics of gauged linear sigma models: new techniques were found for evaluating the partition function of the theory for the case of a sphere, a torus, a hemisphere, and a cylinder. I will focus on the case of the sphere.

The key formula is

$$Z(S^2) = e^{-K},$$

where K is the Kähler potential on the moduli space of the theory.

The Two-Sphere Partition Function

For the mirror variety, the exponentiated Kähler potential has an expression of the form

$$\int_X \Omega_z \wedge \overline{\Omega_z},$$

regarded as a (non-holomorphic) function of the moduli coordinate z. This expression can be evaluated near a large complex structure limit points in terms of periods, and it turns out that the predicted Gromov–Witten invariants can be extracted from the expression.

Thus, a physical argument is for the first time predicting Gromov–Witten invariants directly from the "A-model." We used this to make predictions for Gromov–Witten invariants in some "nonlinear" GLSMs, where those predictions were previously unknown.