GEOMETRY OF COMPLEX PROJECTIVE VARIETIES

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1. Introduction

This is a preliminary lecture note for the algebraic geometry course at the Second International Undergraduate Mathematics Summer School held at Tokyo University, July 29-August 9, 2019.

It is of course impossible to teach algebraic geometry in five 90-minute lectures. Thus this would not be what I will try to do. Indeed, the goal of these lecture notes is to convey to the students a sense of what kind of objects/problems we study in algebraic geometry.

Modern algebraic geometry requires the full machinery of Grothendieck: the language of schemes, sheaf and cohomology, and so on. On the other hand, algebraic geometry deals with polynomials, which are quite explicit. Quite often, explicit computations illustrate the general machinery. So I choose to (and probably have to) follow the low-tech approach and do things in an explicit way, giving hints to a general theory, if possible.

In these lectures, I start with some classical examples of projective varieties, and then whenever I introduce a new concept, I study this concept with these examples in some details. Somewhat surprisingly, one can study these examples and prove interesting theorems rigorously with a minimum amount of prerequisite (one the level of what a good undergraduate student in his/her second or third year has learned), and some hard work. It is my hope that through the course, the students could get a rough idea of some of the basic ways of thinking about a problem in algebraic geometry.

I assume the students are familiar with differentiable manifolds, vector bundles, some basic topology and abstract algebra. But to really understand the lectures, certain level of the mysterious "mathematical maturity" is absolutely essential. To avoid technicalities (more precisely, to make the students feel that they are in their comfort zone of differentiable manifolds), I will work over the complex numbers, although for the most part, I did not use anything special about complex manifold. I will try to make this course as self-contained as possible. But with Google and Wikipedia (or similar websites), I think the students should be able to catch up if there are some minor points that they have not encountered previously.

Some helpful references include: [Har95, Mum76, Rei88]. The book by Harris contains lots of invaluable classical examples and exercises. The book of Reid has the same flavor as this note (I found out this book while preparing the lectures and couldn't agree more with the general philosophy in that book) and Chapter 8 is good to read for students interested in going further. Finally it is always enjoyable and enlightening to read a textbook by a master in the field, especially when the author is a good writer like Mumford.

This is only a preliminary version. Lots of details are left out, some of which might be added later. A place to find the update (if it ever exists) is:
`https://sites.google.com/view/zhiyutian/teaching/summer-2019`.

2. Varieties

Definition 2.1. An affine space of dimension n is the vector space $\mathbb{C}^n$. We denote it by $\mathbb{A}^n$. An affine algebraic set is the subset of $\mathbb{A}^n$ defined by the vanishing locus of a set of polynomials $f_\alpha \in \mathbb{C}[X_1, \ldots, X_n]$.

Definition 2.2. The projective space of dimension n is the set of 1-dimensional vectors of $V = \mathbb{C}^{n+1}$. We denote it by $\mathbb{P}^n$. Sometimes we use $\mathbb{P}(V)$ to emphasize that this comes from a vector space. A projective algebraic set is the vanishing locus of *homogeneous polynomials*.

Example 2.3. Some examples and terminologies.

(1) Linear space.
(2) Hypersurface.
(3) Twisted cubic, rational normal curve.
(4) Nodal and cuspidal rational curve.
(5) Determinantal varieties $M_k(n, m), \mathbb{P}(M_k(n, m))$. Twisted cubic as a determinantal variety.
(6) Grassmannian. $G(2, n), G(2, 4)$

Example 2.4 (Veronese map)**.**

$$v_d : \mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^d V)$$
$$[X_0, \ldots, X_n] \mapsto [X_0^d, X_0^{d-1}X_1, X_0^{d-1}X_2 \ldots, X_n^d]$$

The rational normal curves are the Veronese images of $\mathbb{P}^1$.

Example 2.5. The Veronese surface $v_2(\mathbb{P}^2)$ deserves a special mention.

The space $\mathrm{Sym}^2 V$ can be thought of as symmetric bilinear forms on V^*. So each element is same as a symmetric matrix. The elements of the form $v \otimes v, v \in V$ are precisely the non-zero symmetric matrices of rank 1. Thus $v_2(\mathbb{P}(V))$ can be described as a determinantal variety. In particular, $v_2(\mathbb{P}^2)$ can be described as

$$\mathrm{rk}\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_3 & X_4 \\ X_2 & X_4 & X_5 \end{pmatrix} \leq 1$$

Example 2.6 (Segre map)**.**

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$$
$$([X_0, \ldots, X_n], [Y_0, \ldots, Y_m]) \mapsto [X_0Y_0, \ldots, X_iY_j, \ldots, X_nY_m]$$

This is also an example of determinantal varieties. The typical use of the Segre embedding is to show that a product of projective varieties is projective.

Example 2.7.

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$
$$([X_0, X_1], [Y_0, Y_1]) \mapsto [X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1]$$

The image is the quadric surface $Z_0Z_3 - Z_1Z_2$ in $\mathbb{P}^3$. There are two families of lines (rulings) coming from the two $\mathbb{P}^1$ factors.

Exercise 2.8. Prove that given 3 disjoint lines in $\mathbb{P}^3$, there is a unique quadric surface containing all three of them. Furthermore, they belong to one of the two families.

Example 2.9 (Cone, join)**.** Let $X \subset \mathbb{P}^n \subset \mathbb{P}^{n+1}$ be a projective variety and $v \in \mathbb{P}^{n+1} - \mathbb{P}^n$. The cone over X with vertex v is the subvariety

$$C(X) = \cup_{x \in X} \overline{xv},$$

where $\overline{xv}$ is the line containing x and v.

Let $X, Y \subset \mathbb{P}^n$ be two disjoint projective varieties. The join of X, Y is

$$C(X,Y) = \cup_{x \in X, y \in Y} \overline{xy}.$$

How to prove these are algebraic varieties? One useful thing to keep in mind is the following:

Principle 2.10. Algebraic constructions give algebraic objects.

Exercise 2.11. Prove that $C(X)$ and $C(X,Y)$ are projective varieties by writing down equations using defining equations of X, Y (and maybe some more). Hint: first do the case of $C(X)$. Then treat the case X, Y are in disjoint linear spaces. Finally, $C(X,Y)$ may be reduced to the previous case.

2.1. **Zariski topology.**

Definition 2.12. Zariski topology on a variety: closed subsets are algebraic subsets.

Example 2.13. Closed sets in $\mathbb{A}^1$ are either empty, finite sets or $\mathbb{A}^1$.

Exercise 2.14.

(1) Check that this indeed defines a topology.

(2) Given an affine variety $X \subset \mathbb{A}^n$, and a polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$, define the subset

$$X_f := \{x \in X | f(x) \neq 0\}.$$

Show that such sets form a basis for the Zariski topology. In fact, prove that any open subset is a finite union of such opens. We call them basic open subsets.

(3) Show that X_f is also affine.

Remark 2.15. An affine algebraic set is compact in the Zariski topology by the above exercise. We say it is paracompact in algebraic geometry, probably because we do not want to think of an affine variety as being "compact". We need a different notion for the real "compactness".

Since we work over the complex numbers, one can also endow every algebraic subset with the subspace topology of $\mathbb{A}^n$ or $\mathbb{P}^n$. We will call this the classical/analytic topology.

In the remaining of the notes, we will use Zariski topology unless otherwise specified.

Definition 2.16. A topological space X is irreducible if $X = X_1 \cup X_2$, X_1, X_2 closed subsets of X implies that $X = X_1$ or $X = X_2$.

An algebraic subset is an algebraic variety if it is irreducible (with respect to the Zariski topology).

Definition 2.17. Quasi-projective variety: open subset of a closed subvariety in $\mathbb{P}^n$.

When we say subvariety, we always mean a closed subvariety.

One important usage of Zariski topology is to define the following:

Definition 2.18. Given a variety X, we say a point x is general if it is outside a Zariski closed subset of X (in each situation where we talk about a general point, the subset should be specified. Usually we take it to be the subset whose points satisfies some algebraic conditions).

Example 2.19. For example, we may say that there is a unique line passing through two general points of $\mathbb{P}^n$. This means that for a pair of points $(x, y) \in \mathbb{P}^n \times \mathbb{P}^n$, outside a closed subset of $\mathbb{P}^n \times \mathbb{P}^n$ (exercise: which subset?), there is a unique line passing through x and y.

2.2. **Ideals, regular functions, morphisms.** Let $V \subset \mathbb{A}^n$ be an affine algebraic set. Define *the ideal of* V as

$$I(V) = \{f \in \mathbb{C}[x_0, \ldots, x_n] | f|_V = 0\}$$

The coordinate ring of V is $\mathbb{C}[x_0, \ldots, x_n]/I(V)$.

This is the ring of regular functions on V.

Let $X \subset \mathbb{P}^n$ be a projective algebraic set. Define *the homogeneous ideal of* X as

$$I(X) = \{f \in \mathbb{C}[x_0, \ldots, x_n] | f|_V = 0 \text{ and } f \text{ is homogeneous}\}$$

The homogeneous coordinate ring of X is $\mathbb{C}[x_0, \ldots, x_n]/I(X)$

Theorem 2.20 (Hilbert Nullstellensatz)**.**

$$I(V(I)) = \sqrt{I}.$$

where $\sqrt{I}$ *is the radical of* I*, i.e.* $f \in \sqrt{I}$ *if and only if* $f^n \in \sqrt{I}$ *for some* n*.*

Corollary 2.21.

$$V(I) \subset V(J) \textit{if and only if } \sqrt{J} \subset \sqrt{I}.$$

Corollary 2.22. *There is a one-to-one, inclusion reversing, correspondence between*

(1) $\{$*radical ideals*$\}$ *and* $\{$ *algebraic sets*$\}$,
(2) $\{$*prime ideals*$\}$ *and* $\{$*subvarieties*$\}$.

Corollary 2.23 (Weak Nullstellensatz)**.** *Maximal ideals in* $\mathbb{C}[x_1, \ldots, x_n]$ *are of the form* $(x_1 - a_1, \ldots, x_n - a_n)$.

Definition 2.24 (Open affine cover)**.** Let $U_i \subset \mathbb{P}^n$ be the open subset $X_i \neq 0$. Then $U_i \cong \mathbb{A}^n$. For a projective variety X, $X \cap U_i$ is an affine variety. If $F \in I(X)$, then $f = F(\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}) \in I(X \cap U_i)$

This shows that a projective variety is glued from its affine open subvarieties. There are many ways to construct affine open covers.

Exercise 2.25. Let $X_d \subset \mathbb{P}^n$ be a degree d hypersurface. Prove that $U = \mathbb{P}^n - X_d$ is an affine variety. Prove that any open cover of a quasi-projective variety has a refinement consisting of affine open covers.

Using the affine covers U_i, you can do the following.

Exercise 2.26. Formulate a version of the Nullstellensatz for projective varieties. Note that the ideal generated by all the X_i's already defines the empty subset of $\mathbb{P}^n$.

Definition 2.27 (Morphism)**.** A morphism between affine varieties $f : V \to U$ is given by a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[U] \to \mathbb{C}[V]$. A morphism between quasi-projective is a map which is locally given by morphisms of affine varieties.

Example 2.28 (Projection from a linear space)**.** Let $X \subset \mathbb{P}^n$ be a projective variety, and let $\Lambda \subset \mathbb{P}^n$ be a linear subspace of dimension $n - m - 1$ disjoint from X. Assume that Λ is defined by $X_0 = \ldots = X_m = 0$. There is a morphism $\mathbb{P}^n - \Lambda \to \mathbb{P}^m, [X_0, \ldots, X_n] \mapsto [X_0, \ldots, X_m]$. We can restrict this to X and get the projection from Λ.

Exercise 2.29. Let $C \subset \mathbb{P}^3$ be a twisted cubic, and let $x \in \mathbb{P}^3$ be a point disjoint from C. Show that the image of C under the projection from x is either a nodal or a cuspidal plane cubic.

This might by difficult. If you cannot do the general case, try to work out some special cases. Or instead, work on an easy problem: find a point such that the projection is a nodal (resp. cuspidal) plane cubic.

Example 2.30 (Veronese map)**.**

$$\mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^d V)$$
$$[X_0, \ldots, X_n] \mapsto [X_0^d, X_0^{d-1}X_1, X_0^{d-1}X_2 \ldots, X_n^d]$$

The rational normal curves are the Veronese images of $\mathbb{P}^1$.

Example 2.31 (Segre map)**.**

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$$
$$([X_0, \ldots, X_n], [Y_0, \ldots, Y_m]) \mapsto [X_0Y_0, \ldots, X_iY_j, \ldots, X_nY_m]$$

The typical use of the Segre embedding is to show that a product of projective varieties is projective.

Finally we record the following theorem, which will be used later.

Theorem 2.32 (Chevalley, [Har77, Chap 2, Ex. 3.19])**.** *Let $f : X \to Y$ be a morphism between varieties. Then the image of f is constructible, i.e. it is a finite union of locally closed (=intersection of an open set with a closed set) subset of Y.*

Example 2.33. $f : \mathbb{A}^2 \to \mathbb{A}^2, (x, y) \mapsto (xy, y)$. The image is only constructible.

2.3. **Products, Correspondences, Rational maps.**

Definition 2.34. Let X, Y be affine varieties in $\mathbb{A}^n, \mathbb{A}^m$ defined by polynomials $F_i(x), G_j(y)$. The product $X \times Y$ is defined as the closed subset in $\mathbb{A}^{n+m}$ defined by $F_i = G_j = 0$.

Lemma 2.35. *Let X, Y be irreducible topological spaces. Suppose that the topology on $X \times Y$ is such that for any $x \in X, y \in Y$, the subspace topology on $x \times Y$ and $X \times y$ is the same as that of X and Y. Then $X \times Y$ is irreducible with this topology.*

Exercise 2.36. Prove that $\mathbb{C}[X \times Y] \cong \mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[Y]$. Check that this is the product in the category of quasi-projective varieties, i.e. it satisfies the universal property of a product.

IMPORTANT: The Zariski topology on $X \times Y$ is *NOT* the product topology. The simplest example is to look at $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$.

Definition 2.37 (Product of projective varieties)**.** Let X, Y be projective varieties in $\mathbb{P}^n$ and $\mathbb{P}^m$. Then $X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{nm+n+m}$ is a projective variety (see below for justifications).

Lemma 2.38. *An algebraic set in $\mathbb{P}^n \times \mathbb{P}^m$ is defined by $G_\alpha(X_0, \dots, X_n, Y_0, \dots, Y_m) = 0$ for polynomials homogeneous for X_i and homogeneous for Y_j*

Definition 2.39 (Product of quasi-projective varieties)**.** This is easy, once we know products of projective varieties is projective.

Lemma 2.35 gives

Corollary 2.40. *The product of varieties is again a variety.*

Exercise 2.41. Prove:

(1) Let X be a topological space. X is Hausdorff if and only if the diagonal $\Delta_X \subset X \times X$ is closed, where $X \times X$ is given the product topology.
(2) Let $U \subset \mathbb{A}^n$ be an affine algebraic set. The diagonal $\Delta_U \cong U \subset U \times U$ is closed, where $U \times U \subset \mathbb{A}^{2n}$ is given the Zariski topology.
(3) Let X be a quasi-projective variety. Then the diagonal $\Delta_X \subset X \times X$ is closed, where $X \times X$ is given the Zariski topology.
(4) Let $U, V \subset \mathbb{A}^n$ be affine algebraic sets. Then $U \cap V$ is also an affine algebraic set.
(5) Let X be a quasi-projective variety. Let $U, V \subset X$ be *open* affine subvarieties of X. Then $U \cap V$ is also an open affine subvariety of X.

With this exercise, we can define the ring of regular functions for any quasi-projective varieties. Usually one defines this as the global sections of the structure sheaf. Here we are essentially doing the same thing, using Čech covers, and without mentioning anything sheaf theoretic.

We proceed in several steps.

Exercise 2.42. Let U be an affine algebraic set and $U_i, 1 \leq n$ be an open affine cover of U. Then

$$\mathbb{C}[U] \cong \mathrm{Ker}(\oplus_{i=1}^n \mathbb{C}[U_i] \to \oplus_{1 \leq j \neq k \leq n} \mathbb{C}[U_j \cap U_k]),$$

where the map are the natural restriction maps

Exercise 2.43. Let X be a quasi-projective variety algebraic set and $U_i, 1 \leq n$ be an open affine cover of X. Then

$$\mathbb{C}[X] = \mathrm{Ker}(\oplus_{i=1}^n \mathbb{C}[U_i] \to \oplus_{1 \leq j \neq k \leq n} \mathbb{C}[U_j \cap U_k])$$

is well-defined, which is the ring of regular functions on X.

Remark 2.44. Here the essential thing is that the intersection of two affine open subsets is still affine, on which we understand how to define the ring of regular functions. Clearly, the diagonal being closed is crucial for this approach. In scheme theory terms, such property is called *separated.* For us, very thing is separated since we only consider quasi-projective varieties (algebraic sets).

Exercise 2.45. Let X be a projective variety. Prove that $\mathbb{C}[X] \cong \mathbb{C}$. This should remind you the corresponding statement for holomorphic functions on compact complex manifold.

Proposition 2.46. *Let X be a variety and Y be an affine variety.*

$$Hom(X,Y) = Hom_{\mathbb{C}-alg}(\mathbb{C}[Y], \mathbb{C}[X])$$

Definition 2.47. Let X, Y be quasi-projective varieties. A correspondence from X to Y is a closed algebraic set Z of $X \times Y$.

Example 2.48. Let $f : X \to Y$ be a morphism. The graph Γ_f gives a correspondence in $X \times Y$.

Example 2.49. Consider the *universal lines* $\mathcal{U} \subset G(2,V) \times \mathbb{P}(V)$, defined as the set of pairs $\{(\Lambda.v) \in G(2,V) \times \mathbb{P}(V) | v \in \Lambda\}$. This is a correspondence.

Example 2.50.

$$\{(A, \Lambda) | A : \mathbb{C}^n \to \mathbb{C}^m, \mathrm{rank} A \leq k, \Lambda \subset \ker A, \dim \Lambda = n - k\}$$

This is a correspondence in $\mathbb{P}(M_k(n,m)) \times G(n-k, n)$

Definition 2.51. Let X, Y be varieties. A rational map between X, Y is the equivalence class of morphisms $f : U \to Y$, where U is a Zariski open subset of X. The equivalence relation is

$$(f : U \to Y) \sim (g : V \to Y) \text{ if and only if } f = g : U \cap V \to Y.$$

We denote it by $f : X \dashrightarrow Y$. Given a rational map $f : X \dashrightarrow Y$, we define the graph as the Zariski closure of Γ_f for some $f : U \to Y$. This gives a correspondence from X to Y.

Given two rational maps $f : X \dashrightarrow Y, g : Y \dashrightarrow Z$, if f is dominant, then we can define the composite

$$g \circ f : X \dashrightarrow Z.$$

Example 2.52.

$$f : \mathbb{P}^2 \to \mathbb{P}^2$$

$$[X, Y, Z] \mapsto [\frac{1}{X}, \frac{1}{Y}, \frac{1}{Z}]$$

$f \circ f = id$

Definition 2.53. A *rational function* on a variety X is a regular function defined over some Zariski open subset. Equivalently, it is a rational map $X \dashrightarrow \mathbb{A}^1$.

The *function field* of X is the field of rational functions on X.

2.4. **GAGA principle.** The projective space can be given the structure of a complex manifold. For those who prefer a geometric setting, it is quite safe to think of every variety that we will talk about in the notes as a complex manifold, except for the morphisms between varieties. There is a big difference between holomorphic and algebraic maps between non-projective manifolds. For example, think about the exponential function e^z and polynomials and the Picard theorem(s).

We recall the following Theorem of Chow, which is the first instance of the so-called GAGA principle.

Theorem 2.54 (Chow). *Let $X \subset \mathbb{P}^n$ be an analytic subset. Then X is algebraic, i.e. X is the vanishing locus of some homogeneous polynomials. Any holomorphic maps between projective manifolds is algebraic.*

Remark 2.55. So for complex projective manifolds, we could just say holomorphic instead of algebraic in front of everything. If you are new to algebraic geometry but are comfortable with holomorphic maps between complex manifolds, there is no harm to think of everything as holomorphic, and it is the same thing when dealing with projective varieties.

3. Basic properties

3.1. Dimension and degree.

Definition 3.1 (Dimension)**.** Given a topological space X, we define its dimension to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of distinct irreducible closed subsets of X. The dimension of a variety is the dimension of the underlying topological space.

Example 3.2. $\dim \mathbb{A}^n = n, \dim \mathbb{P}^m = m$. Can you prove this in an elementary way?

Example 3.3 (Dimension of a hypersurface)**.** Let X be an affine algebraic set of dimension n. $f \in \mathbb{C}[X]$. Then $V(f) \subset X$ has dimension at least $n-1$ (Krull's hauptidealsatz).

Lemma 3.4. (1) $\dim(X \times Y) = \dim X + \dim Y$.
(2) *Let X be a variety and U be an open subset of X. Then* $\dim X = \dim U > \dim(X - U)$.

Example 3.5 (Dimension of Grassmannian)**.** $\dim G(r, n) = r(n - r)$.

Exercise 3.6. Determine the dimension of a generic determinantal variety $M_k(n, m)$.

Chevalley's theorem (Theorem 2.32) gives the following.

Lemma 3.7. *Let $f : X \to Y$ be a morphism of varieties.*

$$\dim \overline{f(X)} \leq \dim X.$$

If $\dim X < \dim Y$, the image of f is contained in a proper subvariety of Y.

Example 3.8. Recall the space filling Peano's curve. This lemma is a special feature of algebraic maps.

Example 3.9 (Secant variety)**.** The secant variety of $X \subset \mathbb{P}^n$ is

$$\mathrm{Sec}(X) = \overline{\{z \in \mathbb{P}^n | z \in C(x, y), x \neq y\}}$$

We would like to study the dimension of $\mathrm{Sec}(X)$.

Consider the correspondence

$$\overline{\{(x, y, z) \in (X \times X - \Delta_X) \times \mathbb{P}^n | x \neq y, x, y, z \text{ are colinear}\}}.$$

It has a natural projection to $X \times X$, whose fiber has dimension 1 over the open subset $X \times X - \Delta_X$. Thus the dimension of the correspondence is $2 \dim X + 1$. It follows that $\dim \mathrm{Sec}(X)$, being the image under the third projection, has dimension at most $2 \dim X + 1$.

Exercise 3.10. What is the secant variety of a twisted cubic?

Exercise 3.11. What is the secant variety of a Veronese surface in $\mathbb{P}^5$? Hint: It is not $\mathbb{P}^5$. Thinking in terms of determinantal varieties is helpful. This is a classical example where the inequality is strict.

Exercise 3.12. Study the dimension of $C(X,Y)$ in a similar way as above. That is, find the dimension bound, give examples so that the bound is achieved/not achieved.

Theorem 3.13. *Let $f : X \to Y$ be a morphism of algebraic varieties. Then the function $Y \to \mathbb{Z}, y \mapsto \dim f^{-1}(y)$ is upper-semi-continuous. Equivalently, the set $\{y \in Y|, \dim f^{-1}(y) \geq d\}$ is closed for any d.*

Corollary 3.14. *Let $f : X \to Y$ be a morphism of algebraic varieties.* $\dim X = \dim Y + \dim f^{-1}(y)$ *for y in some non-empty Zariski open subset of Y.*

Definition 3.15. The number $\dim f^{-1}(y)$ in Corollary 3.14 is the *generic fiber dimension of f.*

Next, we show how a simple dimension argument could be used to prove some important results. We consider a homogeneous polynomial in $n+1$ variables of degree d with coefficients in $\mathbb{C}(t)$, $F(X_0, \ldots, X_n) \in \mathbb{C}(t)[X_0, \ldots, X_n]$.

Theorem 3.16 (Tsen-Lang)**.** *There is a non-trivial solution of $F = 0$ in $\mathbb{C}(t)$ if $d \leq n$.*

In modern terms, this says that $\mathbb{C}(t)$ is a C_1-field. A similar argument as below shows that the function field of an algebraic curve over an algebraically closed is C_1. We refer the students to Google to find out the definitions and context, etc. about the following corollary.

Corollary 3.17. *The Brauer group of $\mathbb{C}(t)$ is trivial.*

Proof of Theorem 3.16. After clearing the denominators, we may assume that the coefficients of F are in $\mathbb{C}[t]$. Then $F = 0$ defines a hypersurface $X \subset \mathbb{A}^1 \times \mathbb{P}^n$. A non-trivial solution in $\mathbb{C}(t)$ is the same as a non-trivial solution in $\mathbb{C}[t]$, again by clearing the denominators. Such solutions are in one-to-one correspondence with a morphism $\mathbb{A}^1 \to X$ such that the composition

$$\mathbb{A}^1 \to X \subset \mathbb{A}^1 \times \mathbb{P}^n \to \mathbb{A}^1$$

is the identity. Such a morphism is called a section of $\pi : X \to \mathbb{A}^1$. □

Exercise 3.18. Prove that a section $f : \mathbb{A}^1 \to X \subset \mathbb{A}_t \times \mathbb{P}^n([X_0, \ldots, X_n]), X_i = f_i(t), \deg f_i \leq e$ corresponds to a point in an algebraic set of $\mathbb{A}^{(e+1)(n+1)}$ defined by $de+1$ equations, except the origin $(0, \ldots, 0)$. Thus it has positive dimension (if non-empty) as long as e is large and $d \leq n$. Note that $(0, \ldots, 0)$ is always contained in the algebraic set. Conclude the existence of a morphism $f : \mathbb{A}^1 \to X, X_i = f_i(t), \deg f_i \leq e$.

Exercise 3.19. Write down a homogeneous polynomial in $n+1$ variables of degree $n+1$ with coefficients in $\mathbb{C}(t)$, $F(X_0, \ldots, X_n) \in \mathbb{C}(t)[X_0, \ldots, X_n]$, such that $F = 0$ has only the trivial solution in $\mathbb{C}(t)$.

The followings will be optional. We only discuss them if we have time.

Lemma 3.20 (Positivity of local intersection number)**.**

Corollary 3.21.

$$X \cdot Y \geq |X \cap Y|,$$

if the right hand side is finite.

This positivity is special for algebraic geometry, in contrast to general cases in topology.

Definition 3.22 (Degree of a projective variety)**.**

Theorem 3.23 (Bézout Theorem)**.**

3.2. **Smoothness.**

Definition 3.24 (Zariski tangent space)**.** (1) Affine tangent space.

Let $X \subset \mathbb{A}^n$ be an affine variety and let $f_1, \dots, f_m$ be a set of generators of $I(X)$. Given a point $x \in X$, the tangent space at x is defined to be the affine subspace

$$T_x X = \{v \in T_x\mathbb{A}^n \cong \mathbb{C}^n | \sum_j \frac{\partial f_i}{\partial x_j}(x) v_j = 0, 1 \le i \le m\}.$$

(2) Projective tangent space.

Let $X \subset \mathbb{P}^n$ be a projective variety and let $F_1, \dots, F_m$ be a set of homogeneous generators of $I(X)$. Given a point $x \in X$, the projective tangent space at x is defined to be the projective linear subspace

$$\mathbb{P}(T_x X) = \{v \in \mathbb{P}(T_x\mathbb{P}^n) \cong \mathbb{P}^{n-1} | \sum_j \frac{\partial F_i}{\partial x_j}(x) v_j = 0, 1 \le i \le m\}.$$

(3) Intrinsic tangent space.

The cotangent space at $x \in X$ is defined to be the $\mathbb{C}$-vector space $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of x in an affine neighborhood of x. The tangent space at x is defined to be the dual of $\mathfrak{m}/\mathfrak{m}^2$. For this to make sense, we need to prove that this vector space is independent of the choice of the affine neighborhood.

(4) Regular and singular points.

A point $x \in X$ is a *regular point* if $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim X$. Otherwise it is a *singular point*. A regular point is also called a *smooth point*.

(5) A variety is *regular* if every point is a regular point. Otherwise it is *singular*.

Remark 3.25. One can check that the abstract definition gives the tangent space as a subspace of the tangent space of the affine space, as described in the affine tangent space case.

Example 3.26 (Tangent space to Grassmaniann)**.** $T_\Lambda G \cong \mathrm{Hom}(\Lambda, V/\Lambda)$

Example 3.27. Let X be a smooth projective variety. The projective cone $C(X)$ is singular at the vertex.

Example 3.28. Nodal and cuspidal singularities.

Example 3.29 (Plane curve singularities and knots)**.** We discuss the appearance of knots in the study of plane curve singularities.

Example 3.30 (tangent variety)**.** Let $X \subset \mathbb{P}^n$ be a subvariety. By Theorem 3.32, there is a Zariski open subset U consisting of regular points. Define the tangent variety to X to be

$$\mathrm{Tan}(X) = \overline{\{z | z \in \mathbb{P}(TxX), x \in U\}}$$

Exercise 3.31. (1) $\mathrm{Tan}(X) \subset \mathrm{Sec}(X)$.

(2) $\dim \mathrm{Tan}(X) \leq 2 \dim X$.

Theorem 3.32. *The set of regular points in a variety is non-empty, and Zariski open.*

Corollary 3.33. *Let X be a smooth projective variety of dimension n. Then X can be embedded into $\mathbb{P}^{2n+1}$.*

Proof. Choose an embedding. Then project from a linear space away from $\mathrm{Sec}(X)$. □

Remark 3.34. So there is nothing one can say about a smooth variety X in $\mathbb{P}^n$ if $\mathrm{Codim}(X) > \dim(X)$. On the other hand $\mathrm{Codim}(X) < \dim(X)$, we expect X to be special. Hartshorne made some conjectures.

Exercise 3.35. What is the tangent variety of a twisted cubic? How is this (and the description of its secant variety) related to image of the projection from a point?

Example 3.36 (Dual hypersurface)**.** Let $X \subset \mathbb{P}(V)$ be a hypersurface. The dual hypersurface is the Zariski closure of the map $x \in U \mapsto \mathbb{P}(T_x U) \in \mathbb{P}(V^*)$, where U is the smooth locus of X.

For example, the dual hypersurface of a quadric hypersurface is again a quadric hypersurface.

In general, the dual hypersurface is singular.

Next we discuss the relative notion of smoothness.

Definition 3.37. Let $f : X \to Y$ be a morphism between varieties and $x, y = f(x)$ be smooth points of X, Y. We say f is smooth at x if the induced map on tangent spaces $df : T_x X \to T_y Y$ is surjective.

Remark 3.38. This notion corresponds to the notion of submersion in differential topology.

If f is smooth at x, then it is smooth in a neighborhood of x and the fiber $f^{-1}(f(x))$ is smooth in a neighborhood of x.

Theorem 3.39 (Generic smoothness=Sard's theorem in the algebro-geometric setting)**.** *Let $f : X \to Y$ be a morphism between varieties. Assume that X is smooth. Then there is an open subset $U \subset Y$ such that for any $x \in f^{-1}(U)$, f is smooth at x. Furthermore, U is non-empty if $\overline{Im(f)} = Y$.*

Proof. We give a proof assuming Sard's theorem on the density of regular points, which says that the set of regular values in X is dense. On the other hand, one can show that the set of points in X along which f is not smooth is a closed subvariety W. By the theorem of Chevalley (Theorem 2.32), $f(W)$ is constructible. Putting these two things together, we see that the Zariski closure of $f(W)$ is not the whole Y. Take U to be the complement. □

Remark 3.40. This (and the following corollary) is the only result in these notes that is not true over a general field. It is false in positive characteristic because of morphisms like the Frobenius.

Theorem 3.41 (Bertini's theorem, complex version)**.** *Let $f : X \to \mathbb{P}^n(V)$ be a morphism from a smooth variety X. Then for a general linear space L, $f^{-1}(L)$ is smooth.*

Proof. This proof uses the so-called correspondences, which is a very useful technique in algebraic geometry. Once the correct correspondence is set-up, it is an application of the generic smoothness theorem. We leave it as an exercise. $\square$

Theorem 3.42 (Bertini's theorem, general version, true in any characteristic)**.** *Let $X \subset \mathbb{P}^n(V)$ be a smooth variety. Then for a general linear space L, $L \cap X$ is smooth.*

Proof. Work out the dimension of the following space:

$$\{(x, H) \in X \times \mathbb{P}(V^*) | x \in H, X \cap H \text{ is singular at } x\},$$

by considering the first projection. Then consider the second projection to $\mathbb{P}(V^*)$. It suffices to show that the projection is not surjective. $\square$

3.3. **Properness.**

Definition 3.43 (Proper variety)**.** A morphism $f : X \to Y$ is proper if it is universally closed. That is, for any variety S, the morphism $S \times X \to S \times Y$ maps closed subsets to closed subsets. A quasi-projective variety X is proper if $X \to$ point is proper.

Remark 3.44. This is also an exercise in basic topology: prove that for a compact Hausdorff space X, the map from X to a point is universally closed.

Theorem 3.45. *The projective space is proper.*

This can be done using the elimination theory.

Corollary 3.46. *Any projective variety is proper. Any morphism from a projective variety to a variety is proper.*

Proof. Show that a closed immersion is proper. And the composition of proper morphisms is proper. $\square$

Definition 3.47. Let X be a quasi-projective variety. A resolution of singularities is a birational proper morphism $f : Y \to X$.

Theorem 3.48 (Hironaka)**.** *Resolution of singularities exists over $\mathbb{C}$.*

Example 3.49 (Singularities of a determinantal variety)**.**

Exercise 3.50. Resolution of singularities for determinantal varieties. Prove that Example 2.50 gives a resolution of singularities. Can you construct another one by a similar construction, but instead of looking at kernals, look at the images?

4. Parameter space and moduli

Lemma 4.1 (Yoneda's Lemma)**.** *Let $\mathcal{C}$ be a category and $F : \mathcal{C}^{op} \to (Sets)$ be a (contravariant) functor. Then there is a canonical isomorphism*

$$Hom_{Fun}(h_X, F) \cong F(X),$$

where Fun is the category of (contravariant) functors of $\mathcal{C}$, X an object of $\mathcal{C}$, and h_X is the functor

$$h_X : \mathcal{C}^{op} \to (Sets), Y \mapsto Hom_{\mathcal{C}}(Y, X).$$

In particular,

$$Hom_{Fun}(h_X, h_Y) \cong Hom_{\mathcal{C}}(X, Y).$$

Exercise 4.2. Prove this.

The proof is mostly tautological. But the implication of this lemma is conceptually important:

Corollary 4.3. *The functor $X \mapsto h_X$ is a fully faithful embedding.*

Thus to define an object X in $\mathcal{C}$, it suffices to specify the functor h_X. We say that a functor F is *representable* if it is isomorphic to h_X for some object X, and if this is the case, we also say that X *represents* the functor F (or F is represented by X etc.).

Example 4.4. Consider the functor:

$$\begin{aligned}\mathbb{P}^n : (\text{Varieties}) \to &(\text{Sets}),\\ &X \mapsto \{X \times \mathbb{C}^{n+1} \to L, \text{a surjective map of the trivial vector bundle}\\ &\qquad \text{of rank } n+1 \text{ to a line bundle } L\}.\end{aligned}$$

This functor is represented the projective space $\mathbb{P}^n$.

Exercise 4.5.
1. What is the functor represented by a Grassmannian, $\mathbb{A}^1$, $\mathbb{G}_m$?
2. What is the natural transformation represented the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$

Definition 4.6. Let us look at the first example: the universal hypersurface:

$$\begin{aligned}&\mathcal{U} \subset \mathbb{P}^n \times \mathbb{P}(\mathbb{C}[X_0, \ldots, X_n]_d)\\ &\quad = \{(x, X_d) | x \in X_d, \text{where } X_d \text{ is a hypersurface of degree } d\}\end{aligned}$$

Here the parameter space is $\mathbb{P}(\mathbb{C}[X_0, \ldots, X_n]_d)$, the projective space associated to vector space of degree d homogeneous polynomials.

Definition 4.7 (Universal conic)**.** $\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}^5$ defined by

$$\sum a_{ijk} X_0^j X_1^j X_2^k = 0$$

Exercise 4.8. Consider the second projection $\pi : \mathcal{C} \to \mathbb{P}^5$. Is there a section $\sigma : \mathbb{P}^5 \to \mathcal{C}$(i.e. $\pi \circ \sigma = Id_{\mathbb{P}^5}$)? Is there a section $\sigma : U \to \mathcal{C}|U$ if we restrict to a Zariski open subset U?

Definition 4.9 (Fano variety(scheme) of lines)**.** Let $X \subset \mathbb{P}^n$ be a projective variety. The the Fano scheme/variety of lines on X is the scheme/variety $F(X)$ which represents the functor:

$$\begin{aligned}F(X) : (\text{Varieties}) \to &(\text{Sets})\\ &S \mapsto \{\mathcal{L} \subset \mathbb{P}^n \times S, \text{ is a family of lines.}\}\end{aligned}$$

This is not very precise, since we have to say what does "a family of lines" mean, but hopefully you get the idea.

Exercise 4.10. Show that the functor $F(X)$ is a subfunctor of the functor represented by the Grassmaniann $G(2, V)$.

Example 4.11. We provide a concrete description of the Fano scheme of lines on a smooth quadric surface here. The Grassmannian $G(2,4)$ is embedded in $\mathbb{P}^5$ as a quadric hypersurface via the Plücker embedding. The two families of lines corresponds to two conics. Maybe it is best to leave this as an exercise. One should at least try to write down explicitly the equations and check that this is the case.

Exercise 4.12. One can think about this problem in a somewhat cheating way (*cf.* Exercise 2.8). Choose three lines in one ruling of the quadric surface. Prove that a line in $\mathbb{P}^3$ belongs to the other ruling of the quadric surface if and only if it intersects all of the three lines. Also prove that the lines intersecting a fixed line is the intersection of $G(2,4)$ with a hyperplane in $\mathbb{P}^5$.

Exercise 4.13. What is the number of lines in $\mathbb{P}^3$ that intersects 4 general lines? What should general mean in this case?

Example 4.14 (Universal family of Fano varieties of lines on the universal hypersurface)**.** Write down the universal family of lines on the universal hypersurface.

Exercise 4.15 (Realization as the zero locus of a section of a vector bundle when X is a hypersurface)**.** On the Grassmannian $G(2,V)$, there is a tautological rank two vector bundle, which fits into the following short exact sequence of vector bundles on $G(2,V)$:

$$0 \to S \to V \to Q \to 0,$$

where V is the trivial vector bundle whose fiber over each point is the vector space V, S is the subbundle whose fiber over each point is the corresponding 2-dimensional sub-vector space, and Q is the quotient bundle. Show that each degree d hypersurface X_d gives rise to a section of the vector bundle $Sym^d S^*$, and $F(X_d)$ (at least as a set) is the zero locus of the section.

Exercise 4.16. In this exercise, we study some basic properties of the Fano scheme. Let X_d be a hypersurface of degree d in $\mathbb{P}^n$.

(1) Show that if $2d-3-n>0$, then $F(X_d)$ is empty for X_d general.
(2) Let $x \in X_d$ be a point. Prove that the parameter space of lines in X_d containing x can be described as the common zero locus of some homogeneous polynomials of degree $1,2,3,\dots,d$ in $\mathbb{P}^{n-1}$ (some of these polynomials could be 0). In particular, if $d \leq n-1$, there there is a line passing through every point of X_d, for every X_d. Hint: you need to write down the lines in $\mathbb{P}^n$ passing through x and look for conditions that will force the line to be contained in X_d.
(3) Show that if $d \geq n$, then for a general X_d and a general point $x \in X_d$, there is no line in X_d passing through x.
(4) If you feel energetic, prove that if $2d-3-n \leq 0$, and X is general, $F(X_d)$ is smooth of dimension $n-2d+3$. If not, try to prove the following: one can reduce this statement to the case $2d-3-n=0$. Then prove it for $n=d=3$.

All statements in this exercise remain true over any field. With what we have learned so far, we can prove the last part over $\mathbb{C}$, but it requires some hard work.

5. 27 LINES AND BEYOND

R. Donagi, R. Smith, The structure of the Prym map, P. 27:

Wake an algebraic geometer in the dead of the night, whispering "27". Chances are, he will respond: "lines on a cubic surface".

We will use what we have learned so far to prove the following theorem.

Theorem 5.1. *Every smooth cubic surface contains* 27 *lines.*

Lemma 5.2. *Every smooth cubic hypersurface has the same number of lines.*

Exercise 5.3. (1) Find out the 27 lines on the Fermat cubic $X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0$.
(2) Given a general point x in a smooth cubic surface, what is the number of conics passing through x?

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