Introduction to representation theory of real reductive Lie groups and branching problems*

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Short description of the course:

This course gives an introduction to infinite-dimensional representations of real reductive Lie groups such as $GL(n, \mathbb{R})$ by geometric and analytic methods.

I begin with some basic concepts and techniques on real reductive Lie groups, their representations, and global analysis via representation theory, with a number of classical examples.

If time permits, I would discuss some recent developments on branching problems asking "how irreducible representations of groups behave/decompose when restricted to subgroups".

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Lecture 1.

View "representation theory"
A. **from outside**: various approaches and interactions with other areas of mathematics
B. **from inside**: analysis and synthesis

—How are things built up from smallest objects?
—What are smallest objects?

- Building blocks:
—irreducible representations (4th Lecture)
—simple Lie groups (2nd Lecture)
—simple Lie algebras (2nd Lecture)
- Decomposition into irreducibles
Two important cases from representation theory:
  - Induction $\cdots$ e.g. global analysis on homogeneous spaces
  - Restriction $\cdots$ e.g. tensor product

Back to viewpoint A (from outside).

Various examples of classical analysis problems interpreted as special cases of the general problem of irreducible decompositions of (unitary) representations.

Example: three viewpoints of spherical harmonics
—global analysis for Laplacian
—induced representations (analysis on homogeneous spaces)
—restriction of reps (conformal geometry *v.s.* Riemannian geometry)

Reference: A part of the first lecture will be based on the perspectives advocated in the following papers:

T. Kobayashi, Theory of discrete decomposable branching laws of unitary representations of semisimple Lie groups and some applications, Sugaku Expositions **18** (2005), Amer. Math. Soc., 1–37.

(Translated from the original article in Japanese)

T. Kobayashi, Global analysis by hidden symmetry, In: Representation Theory, Number Theory, and Invariant Theory, Progress in Mathematics, vol. **323** (2017), pp. 359–397.

Lecture 2. Reductive groups–examples and structure theory

The second lecture focuses on

real reductive Lie groups (and reductive Lie algebras).

We plan to discuss
- why are they basic?
- what is the definition?
- various examples;
- structure theorems;

and occasionally some advanced topics by presenting "new way of thinkings".

Why are "reductive Lie groups/algebras" basic?
We may apply the philosophy "analysis and synthesis" to Lie algebras.

Definition 2.1. A Lie algebras $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is not abelian and does not contain ideals other than 0 and $\mathfrak{g}$.

Classification theory of simple Lie algebras over $\mathbb{C}$ or $\mathbb{R}$.
–Killing, Cartan (1894 over $\mathbb{C}$, 1914 over $\mathbb{R}$)
–invariants (root system, Dynkin diagram, Satake diagram ([He, Kn])
–uniform construction (Serre, [Hu])

Remark 2.2. Classification of Lie algebras (of finite-dimension) is not so easy. In fact, a description of consecutive extensions of abelian Lie algebras is nontrivial. For example, classification of nilpotent Lie algebras is known only for low dimensional cases.

Slightly more general than the concept of "simple Lie algebras" we also introduce the definitions of semisimple or reductive Lie algebras, and explain by examples a reason why we use "reductive" rather than simple Lie algebras.

Examples of real reductive Lie groups.

$$GL(n,\mathbb{K}) \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \text{or } \mathbb{H} \text{ (quaternionic number field)},$$

$$O(p,q) = \left\{ g \in GL(p+q,\mathbb{R}) : {}^t g \begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix} g = \begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix} \right\},$$

$$Sp(n,\mathbb{R}) = \left\{ g \in GL(2n,\mathbb{R}) : {}^t g \begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix} g = \begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix} \right\}.$$

If time permits, I explain some structure theory of real reductive Lie groups such as Cartan decomposition and parabolic subgroups or spend some time on some abstract theory (e.g. Hilbert's 5th problem).

Structure of real reductive groups.

$$\begin{aligned} &\text{(local)} \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \\ &\text{(global)} \quad G = K \exp \mathfrak{p}. \end{aligned}$$

Polar decomposition

$$G = KAK$$

Iwasawa decomposition

$$G = KAN$$

Minimal parabolic subgroup

$$P = MAN, \quad M = \text{centralizer of } A \text{ in } K$$

Theorem 2.3. *(an affirmative solution to Hilbert's 5-th problem) The concept of C^k Lie groups is essentially the same.*

Philosophy: algebraic structure raises topological assumptions to analytic results.

cf. Exotic spheres (Kervarie-Milnor)

Exercise : check the above philosophy by a one-dimensional case.

Another example of this philosophy is:

Theorem 2.4 (von Neumann–Cartan)**.** *Any closed subgroup of $GL(N, \mathbb{R})$ carries a natural Lie group structure.*

References for the 2nd lecture:

Textbooks:

[He] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces. Graduate Studies in Mathematics, **34**. American Mathematical Society, Providence, RI, 2001. xxvi+641 pp. ISBN: 0-8218-2848-7

[Hu] J. Humphreys, Introduction to Lie Algebras and Representation Theory. Graduate Texts in Mathematics, **9**. Springer-Verlag, New York-Berlin, 1972. xii+169 pp.

[Kn] A. Knapp, Lie Groups beyond an Introduction. Second edition. Progress in Mathematics, **140**. Birkhäuser Boston, Inc., Boston, MA, 2002. xviii+812 pp. ISBN: 0-8176-4259-5 22-01.

[KO] T. Kobayashi, T. Oshima, Lie groups and Representation Theory, Iwanami 2005, 638pp. ISBN 4-00-006412-9.

Lecture 3. Infinite-dimensional irreducible representation of $SL(2,\mathbb{R})$ — viewpoint of branching laws

The set of equivalence classes of irreducible unitary representations of a group G is called the *unitary dual*, and is denoted by $\widehat{G}$. The building blocks of unitary representations of (algebraic) Lie groups are the unitary dual of simple Lie groups such as $SL(n,\mathbb{R})$. For noncompact simple Lie group G, the unitary dual $\widehat{G}$ contains

• a continuous family of irreducible unitary representations (e.g. principal series representations, complementary series representations),

• a countable family of irreducible unitary representations (e.g. discrete series representations).

The goal of 3rd lecture is to give an **analytic proof** (rather than the usual **algebraic proof**) for the irreducibility of any unitary spherical principal series representation of $G = SL(2,\mathbb{R})$. The proof suggests an intimate connection with various areas in mathematics (outside representation theory), and also with a growing area such as **branching problems** (inside representation theory).

We shall see that the irreducibility in Theorem 3.1 is delicate because a unitary non-spherical principal series is not always irreducible (Remark 3.2 below).

Theorem 3.1. *Let* $\sigma_\lambda(g) := \left|(0,1)g\begin{pmatrix}0\\1\end{pmatrix}\right|^{-\lambda}$ *be a one-dimensional representation of* P *(the group of lower triangular matrices). Then* $L^2\text{-}\mathrm{Ind}_P^G(\sigma_\lambda)$ *is an irreducible unitary representation of* G *for all pure imaginary* λ*.*

A usual proof is based on algebraic techniques ([Ba47], [V]).

In contrast, the strategy of our analytic proof ([KO, Chap11]) uses the restriction to subgroups ("branching laws"). There are three typical one-dimensional subgroups of G:

$$\begin{aligned} N &= \text{abelian subgroup consisting of unipotent elements.}\\ A &= \text{abelian subgroup consisting of hyperbolic elements.}\\ K &= \text{abelian subgroup consisting of elliptic elements.}\end{aligned}$$

Sketch of proof. Suppose W is a closed G-invariant subspace of $L^2\text{-}\mathrm{Ind}_P^G(\sigma_\lambda)$. Obviously,

(1) W is N-invariant,

(2) W is A-invariant,

(3) W is K-invariant.

The condition (1) shows that W is a Wiener subspace in $L^2(\mathbb{R})$, that is, W is a translation invariant closed subspace. This means that W is the Fourier transform of $L^2(E)$ for some measurable set E in $\mathbb{R}$. Then the condition (2) shows that E is either empty, $\mathbb{R}_{>0}$, $\mathbb{R}_{<0}$ or $\mathbb{R}$ up to measure zero set. Finally, we define a natural unitary isomorphism between $L^2(\mathbb{R})$ and $L^2(S^1)$, and the condition (3) shows that E cannot be $\mathbb{R}_{>0}$, $\mathbb{R}_{<0}$, whence $L^2\text{-}\mathrm{Ind}_P^G(\sigma_\lambda)$ is irreducible.

Remark 3.2. There is a similar family of representations with continuous parameter $\lambda \in \mathbb{C}$, called the non-spherical principal unitary representation $L^2\text{-}\mathrm{Ind}_P^G(\sigma_\lambda \otimes \mathrm{sgn})$ which are generically irreducible. However, it is not irreducible at $\lambda = 0$, and splits into the direct sum of two irreducible unitary representations of $G = SL(2, \mathbb{R})$. In fact, the Hardy space is an irreducible submodule at $\lambda = 0$, which corresponds to $E = \mathbb{R}_{>0}$ in the above proof.

References for the third lecture:

[Ba47] V. Bargmann, Irreducible unitary representations of the Lorentz group, Ann. Math., **48**,(1947), 568–640.

[KO] T. Kobayashi, T. Oshima, Lie groups and Representation Theory, Iwanami 2005, 638 pp. ISBN 4-00-006412-9.

[V] D. Vogan, Jr. Representations of Real Reductive Lie Groups, 1981, 754 pp., Birkhäuser, ISBN: 3-7643-3037-6.

Lecture 4. Classification problems of irreducible representations of reductive groups

I plan to explain

- equivariant fiber bundles
- induced representations

and then discuss the current status of the classification theory of irreducible representations together with various approaches:

A irreducible finite-dimensional representations;

B irreducible (infinite-dimensional) admissible representations;

C irreducible unitary representations of real reductive groups such as $GL(n, \mathbb{R})$.

For A, the classification theory is classical, known as the *Cartan-Weyl highest weight theory*.

- invariants

—highest weight (Cartan-Weyl)

- construction

—use complex geometry (Borel-Weil)

—use "universality" (Verma)

For B, there are three approaches:

- analytic approach (an estimate of matrix coefficients —Langlands)
- algebraic approach (Lie algebra cohomology —Vogan)
- geometric approach ($\mathcal{D}$-modules)

For C, the problems are still open, although there have been extensive and important progresses over 70 years.

Lecture 5. Further topics

The fifth lecture contains some advanced topics but I try to keep the lecture to be accessible to the beginners as well. I also plan to mention some cutting-edge of this active area.

We begin with the following.

Naive Question 5.1. Is a vector space V controlled "well" by a group G?

For infinite-dimensional representation V, obstructions for "good control" could be:

a. continuously many irreducibles may occur in V.

b. irreducible representations might be of infinite dimension.

c. multiplicity might be infinite.

Actually, both a and b are harmless, whereas c is serious. Thus, we may formulate Naive Question 5.1 into rigorous problems as follows:

Basic Question 5.2. Given a representation (π, V) of G,

(1) when is the multiplicity $m(\sigma) < \infty$ for all irreducible σ?

(2) when is the multiplicity $m(\sigma)$ uniformly bounded with respect to σ?

Here, for $\sigma \in \mathrm{Irr}(G)$, $m(\sigma) \in \mathbb{N} \cup \{\infty\}$ stands for the multiplicity of σ occurring in V (there are subtle problems to define $m(\sigma)$ when $\dim V = \infty$, but we postpone it for now).

(1) (even better, (2)) will give us a nice framework for detailed study of representations.

Typical Settings for (π, V) in Basic Question 5.2 are as follows:

(global analysis—induction) Given a G-space X, consider the regular representation on $V :=$ the space of functions on X.

(branching problem—restriction) Given an irreducible representation π of $\widetilde{G}$ and its subgroup G, consider the restriction $\pi|_G$ as a representation of G.

To have a better understanding on these two settings, we consider basic questions for induction and restriction. In what follows, we suppose that $G' \subset G$ are pair of real reductive Lie groups.

Induction Analysis on homogeneous spaces G/G' (1950–; long history, but still developing)

There is a natural differential operator (*e.g.* Laplacian) on the homogeneous space G/G' as it carries a G-invariant pseudo-Riemannian structure.

Typical problems in global analysis include:

• Construct eigenfunctions for (natural) differential operators on G/G'.

• Expand arbitrary functions into eigenfunctions.

Let $\mathbf{1}$ denote the trivial one-dimensional representation of G'. Then we have natural isomorphisms

global analysis		induction of representations
$C^\infty(G/G')$	$\simeq$	$C^\infty\text{-}\mathrm{Ind}_{G'}^{G}(\mathbf{1})$
$L^2(G/G')$	$\simeq$	$L^2\text{-}\mathrm{Ind}_{G'}^{G}(\mathbf{1})$

Then we can relate these problems in global analysis with a problem in representation theory of G, such as

• Plancherel-type theorem : irreducible decomposition of the regular representation of the group G on the Hilbert space $L^2(G/G')$.

The relationship is built on a variation of Schur's lemma.

Restriction Branching problem from G to the subgroup G'.

This is a an active and new area, and a systematic study has been started relatively recently, see recent books [SB2015, SB2016, SB2018], and [K-2015] for a program on branching problems of infinite-dimensional representations.

We consider the following setting. Let π be an irreducible representation of G. The restriction $\pi|_{G'}$ is regarded as a representation of the subgroup G', which is no more irreducible in general.

The first important problem is to establish the general theory about the behavior of the restriction $\pi|_{G'}$ (*e.g.* detect whether spectrum is discrete or continuous; multiplicity is finite or infinite, *etc.*) The general theory for spectrum is discussed in [K-1998], and that for multiplicities are explained in Theorem 5.6 and Theorem 5.7 below.

Second, we wish to understand the restriction $\pi|_{G'}$ in a more concrete way. For this, let τ be another irreducible rep of G'. Typical (concrete) problems for restriction include

• Determine when $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \neq \{0\}$;

• (branching law) Decompose $\pi|_{G'}$ into irreducible representations of the subgroup G'.

• Construct intertwining operators (*symmetry breaking operators*) from π to τ.

The second one requires π to be unitary, but the first and third one not.

General theory on multiplicities for induction and restriction.

We now explain the solutions to Basic Question 5.2 for induction and restriction, see [KO-2013] and [K-2014], respectively, for the original articles.

Theorem 5.3. *(global analysis) The following two conditions are equivalent:*

(i) *(representation theory) The space* $\mathrm{Hom}_G(\pi, C^\infty(G/G'))$ *of intertwining operators is finite-dimensional for all* $\pi \in \mathrm{Irr}(G)$.

(ii) *(geometry: real spherical) A minimal parabolic subgroup P has an open orbit in G/G', that is, G/G' is a real spherical manifold.*

Theorem 5.4. *(global analysis) The following two conditions are equivalent:*

(i) *(representation theory) The dimension of* $\mathrm{Hom}_G(\pi, C^\infty(G/G'))$ *is uniformly bounded with respect to* $\pi \in \mathrm{Irr}(G)$.

(ii) *(complex geometry) $B_{\mathbb{C}}$ has an open orbit in $G_{\mathbb{C}}/G'_{\mathbb{C}}$, that is, $G_{\mathbb{C}}/G'_{\mathbb{C}}$ is spherical.*

These two theorems give an answer to Basic Question 5.2 when

$$V = \text{the space of functions on a homogeneous space } G/G'.$$

Remark 5.5. One can extend Theorems 5.3 - 5.4 to the sections of G-equivariant vector bundles of finite rank, and also to non-reductive subgroups G'.

We now give an answer to Basic Question 5.2 in the setting of the restriction, that is,

$$V = \text{ the representation space of a larger group.}$$

Theorem 5.6. *(restriction) The following two conditions are equivalent:*

(i) *(representation theory)* $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ *is finite-dimensional for all* $\pi \in \mathrm{Irr}(G)$ *and* $\tau \in \mathrm{Irr}(G')$.

(ii) *(geometry) A minimal parabolic subgroup P' of G' has an open orbit in G/P, that is,* $(G \times G')/\mathrm{diag}(G')$ *is real spherical.*

Theorem 5.7. *(restriction) The following two conditions are equivalent:*

(i) *(representation theory) The dimension of* $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ *is uniformly bounded with respect to* $\pi \in \mathrm{Irr}(G)$ *and* $\tau \in \mathrm{Irr}(G')$.

(ii) *(complex geometry) A Borel subgroup $B'_{\mathbb{C}}$ of $G'_{\mathbb{C}}$ has an open orbit in $G_{\mathbb{C}}/B_{\mathbb{C}}$, that is, $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\mathrm{diag}(G'_{\mathbb{C}})$ is real spherical.*

Theorems 5.3 - 5.7 single out *nice settings* in which we could expect detailed study on

- (induction) global analysis by using representation theory,
- (restriction) understanding of symmetry breaking.

Some special cases have been studied extensively, and some others are new. Here are some settings that arise from the criteria in Theorems 5.3 - 5.7.

Example 5.8. (1) Reduction symmetric spaces always satisfy the criterion in Theorem 5.4 (cf. Plancherel-type formula, [D-1998]).

(2) Whittaker models (the criterion in Theorem 5.4 is satisfied if G is a quasi-split group).

(3) Tensor product representations have finite-multiplicities when $G = O(n,1)$, as is shown by Theorem 5.6. The classification of the pairs (G, G') satisfying the criterion in Theorem 5.6 is given in [KM-2015].

(4) The geometry (ii) in Theorem 5.7 singles out the pair, (GL_n, GL_{n-1}) and (O_n, O_{n-1}), cf. the Gan–Gross–Prasad Conjecture [GP-1992][KS-2018]).

Theorem 5.3 suggests a construction of eigenfunctions for invariant differential operators on real spherical manifolds.

Let (σ, V) be a finite-dimensional representation of a parabolic subgroup P of G, $\mathcal{V} = G \times_P V$ the homogeneous vector bundle over the real flag variety G/P, and $\mathcal{V}^*_{2\rho}$ the dualizing bundle (see [SB2015], Chapter 3).

Theorem 5.9. (generalized Poisson transform [K-1992]) *Suppose w is an H-invariant element of $\Gamma(X, \mathcal{V}^*_{2\rho})$. Then $(T_w f)(g) := (\pi(g^{-1})f, w)$ induces a G-intertwining operator $T_w\colon \mathrm{Ind}_P^G(\sigma) \to C^\infty(G/H)$. In particular, the image satisfies a system of partial differential equations*

$$D_z \phi = \lambda_\sigma(z)\phi \quad \text{for all } z \in Z(\mathfrak{g}).$$

Here, $Z(\mathfrak{g})$ is the center of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$, D_z is a G-invariant differential operator on G/H induced by $z \in Z(\mathfrak{g}_{\mathbb{C}})$, and $\lambda_\sigma(z)$ is the scalar determined by σ.

Example 5.10. $G = SL(2,\mathbb{R})$ and $K = SO(2)$. Keep notation as in the 3rd lecture. Define w by

$$w(g) = \left\| g \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|^{-\lambda-2}.$$

Let $\mathcal{H}$ be the upper half-plane identified with the upper-half plane G/K. Then

$$T_w \colon \operatorname{Ind}_P^G(\sigma_\lambda) \to C^\infty(G/K)$$

is written in the coordinate as

$$T_w \colon C_c^\infty(\mathbb{R}) \to C^\infty(\mathcal{H}), \quad f \mapsto \int_{\mathbb{R}} f(t)(y/(x-t)^2 + y^2)^{\lambda/2+1} dt$$

is the twisted Poisson transform, and the image satisfies

$$\Delta g = \frac{\lambda(\lambda+2)}{4} g$$

where Δ is the hyperbolic Laplacian $y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. Furthermore, T_w is injective if $\operatorname{Re}\lambda \geq -1$.

The "converse" of T_w is given as "boundary maps", which yield a short proof of Casselman's subrepresentation theorem (cf. [W-1988]) as well as the proof for (ii)$\to$(i) in Theorem 5.3. See [KO-2013] for further perspectives on this topic.

Finally, concerning symmetry breaking operators for the restriction, we mention recent development in connection with conformal geometry. The model space for conformally symmetry breaking operators for hypersurfaces is given as a pair (S^n, S^{n-1}), and its pair of conformal groups arises from the criterion in Theorem 5.7. The construction of all conformal symmetry breaking operators has been recently accomplished in a series of the books [SB2015, SB2016, SB2018].

References on the 5th Lecture and for some further readings:

[D-1998] P. Delorme, Formule de Plancherel pour les espaces symétriques réductifs, Ann. Math. **147**, (1998), pp.417–452.

[GP-1992] B.H.Gross and D.Prasad, On the decomposition of SO_n when restricted to SO_{n-1}, Canad. J. Math. **44** (1992), pp.974-1002.

[K-1992] T.Kobayashi, Singular Unitary Representations and Discrete Series for Indefinite Stiefel Manifolds $U(p,q;F)/U(p-m,q;F)$, Mem. Amer.

Math. Soc. **462**, Amer. Math. Soc., 1992. v+106 pages, ISBN-13: 978-0-8218-2524-2.

[K-1998] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}\lambda)$ with respect to reductive subgroups. II. Micro-local analysis and asymptotic K-support. Ann. of Math. **147** (1998), no. 3, 709–729.

[K-2015] T. Kobayashi. A program for branching problems in the representation theory of real reductive groups. Progress in Mathematics **312**, pages 277-322. Birkhäuser, 2015.

[KO-2013] T. Kobayashi and T. Oshima. Finite multiplicity theorems for induction and restriction. Advances in Mathematics, **248**, (2013), pp.921-944.

[K-2014] T. Kobayashi. Shintani functions, real spherical manifolds, and symmetry breaking operators. Developments in Mathematics, **37**, pages 127-159, 2014.

[KM-2014] T. Kobayashi and T. Matsuki. Classification of finite-multiplicity symmetric pairs. Transformation Groups, **19**, (2014), pp.457-493.

[KS-2018] T.Kobayashi and B.Speh, Symmetry breaking for orthogonal groups and a conjecture by B. Gross and D. Prasad. In: Geometric Aspects of the Trace Formula, Simons Symposium on the Trace Formula, pages 245-266. Springer, Cham, 2018.

[W-1988] Nolan R. Wallach, Real Reductive Groups. I, Pure and Applied Mathematics **132**, Academic Press, 1988.

Books on Symmetry Breaking:

[SB2015] T. Kobayashi, B. Speh, Symmetry Breaking for Representations of Rank One Orthogonal Groups, Memoirs of American Mathematical Society, **238**, (2015), vi+112 pp., ISBN: 978-1-4704-1922-6.

[SB2016] T. Kobayashi, T. Kubo, M. Pevzner, Conformal Symmetry Breaking Operators for Differential Forms on Spheres, Lecture Notes in Mathematics, **2170**, (2016), ix+192 pp. Softcover ISBN: 978-981-10-2656-0. eBook ISBN: 978-981-10-2657-7.

[SB2018] T. Kobayashi, B. Speh, Symmetry Breaking for Representations of Rank One Orthogonal Groups II, Lecture Notes in Mathematics, **2234**, (2018), xv+342 pp., eBook ISBN: 978-981-13-2901-2.