Introduction to operator algebras and their applications to mathematical physics

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1 Introduction

Our aim here is to compare two mathematical approaches to chiral 2-dimensional conformal field theory. Conformal field theory is a kind of quantum field theory and related to many different topics in mathematics. It has attracted much attention of many researchers in mathematics. In this note, we present two mathematical approaches to chiral conformal field theory and show a relation between them. Our emphasis is on the operator algebraic approach based on functional analysis, and we start with basics of operator algebra theory.

Reference [19] is a longer version of this note, and [20] is a shorter version.

2 Basics of Operator Algebras

We prepare some basic facts about operator algebras which are necessary for studying conformal field theory, which was first introduced in [2]. We do not include proofs. As standard references, we list the textbook [30].

2.1 C^*-algebras and von Neumann algebras

We provide some minimal basics on theory of operator algebras. For simplicity, we assume all Hilbert spaces appearing in the text are separable. Our convention for notations is as follows. We use $A, B, \dots, M, N, \dots$ for operator algebras, $a, b, \dots, u, v, \dots, x, y, z$ for operators, $H, K, \dots$ for Hilbert spaces and $\xi, \eta, \dots$ for vectors in Hilbert spaces.

Let H be a complex Hilbert space and $B(H)$ be the set of all bounded linear operators on H. We have a natural $*$-operation $x \mapsto x^*$ on $B(H)$. Here we need two topologies on this set as follows.

Definition 2.1 (1) The *norm topology* on $B(H)$ is induced by the operator norm $\|x\| = \sup_{\|\xi\|\le 1} \|x\xi\|$.

(2) We define convergence $x_i \to x$ in the *strong operator topology* when we have $x_i\xi \to x\xi$ for all $\xi \in H$.

Note that the *strong* operator topology is *weaker* than the norm topology. This is because we have another topology called the weak operator topology, which is weaker than the strong operator topology. The norm convergence is uniform convergence on the unit ball of the Hilbert space and the strong operator convergence is pointwise convergence on the Hilbert space.

Definition 2.2 (1) Let M be a subalgebra of $B(H)$ which is closed in the $*$-operation and contains the identity operator I. We say M is a *von Neumann algebra* if M is closed in the strong operator topology.

(2) Let A be a subalgebra of $B(H)$ which is closed in the $*$-operation. We say A is a *C^*-algebra* if A is closed in the norm topology.

By this definition, a von Neumann algebra is automatically a C^*-algebra, but a von Neumann algebra is quite different from "ordinary" C^*-algebras, so we often think that operator algebras have two classes, von Neumann algebras and C^*-algebras.

A commutative C^*-algebra containing the multiplicative unit is isomorphic to $C(X)$, where X is a compact Hausdorff space and $C(X)$ means the algebra of all complex-valued continuous functions. A commutative von Neumann algebra is isomorphic to $L^\infty(X,\mu)$, where (X,μ) is a measure space.

Easy examples are as follows.

Example 2.3 Let H be $L^2([0,1])$. The polynomial algebra $\mathbb{C}[x]$ acts on H by left multiplication. The image of this representation is a $*$-subalgebra of $B(H)$. (The $*$-operation is given by taking the complex conjugate.) Its norm closure is isomorphic to $C([0,1])$ and its closure in the strong operator topology is isomorphic to $L^\infty([0,1])$.

If a C^*-algebra is finite dimensional, then it is also a von Neumann algebra, and it is isomorphic to $\bigoplus_{j=1}^k M_{n_j}(\mathbb{C})$, where $M_n(\mathbb{C})$ is the $n\times n$-matrix algebra.

Definition 2.4 For $X \subset B(H)$, we set

$$X' = \{y \in B(H) \mid xy = yx \text{ for all } x \in X\}.$$

We call X' the *commutant* of X.

We have the following proposition for von Neumann algebras.

Proposition 2.5 *Let M be a subalgebra of $B(H)$ closed under the $*$-operation and containing I. Then the double commutant M'' is equal to the closure of M in the strong operator topology.*

Note that taking the commutant is a purely algebraic operation, but the above Proposition says it contains information on the topology.

For von Neumann algebras $M \subset B(H)$ and $N \subset \mathcal{B}(K)$, we have natural operations of the direct sum $M \oplus N \subset B(H \oplus K)$ and the tensor product $M \otimes N \subset B(H \otimes K)$. We have the following proposition.

Proposition 2.6 *The following conditions are equivalent for a von Neumann algebra M.*

(1) The von Neumann algebra M is not isomorphic to the direct sum of two von Neumann algebras.

(2) The center $M \cap M'$ of M is $\mathbb{C}I$.

(3) Any two-sided ideal of M closed in the strong operator topology is equal to 0 *or M.*

A natural name for such a von Neumann algebra would be a simple von Neumann algebra, but for a historic reason, this name is not used and such a von Neumann algebra is called a *factor* instead.

2.2 Factors of types I, II and III

The matrix algebra $M_n(\mathbb{C})$ is a factor and the algebra $B(H)$ is also a factor. The former is called a factor of type I_n, and the latter is called a factor of type I_∞ if H is infinite dimensional. We introduce another example of a factor.

Example 2.7 For $x \in M_2(\mathbb{C})\otimes\cdots\otimes M_2(\mathbb{C})$, we consider the embedding $x \mapsto x\otimes I_2 \in M_2(\mathbb{C})\otimes\cdots\otimes M_2(\mathbb{C})\otimes M_2(\mathbb{C})$, where I_2 is the identity matrix in $M_2(\mathbb{C})$. We identify $M_2(\mathbb{C})\otimes\cdots\otimes M_2(\mathbb{C})$ with $M_{2^k}(\mathbb{C})$, where k is the number of the factorial components $M_2(\mathbb{C})$ so that the above embedding is compatible with this identification. Let tr be the usual trace Tr on $M_{2^k}(\mathbb{C})$ divided by 2^k. Then this tr is compatible with the embedding $M_{2^k}(\mathbb{C})$ into $M_{2^{k+1}}(\mathbb{C})$. Let A be the increasing union of $M_{2^k}(\mathbb{C})$ with respect to this embedding. This is a $*$-algebra and the linear functional tr is well-defined on A.

Setting $(x, y) = \mathrm{tr}(y^*x)$ for $x, y \in A$, we make A a pre-Hilbert space. (We use a convention that an inner product is linear in the first variable, which is usual in mathematics, but different from the standard convention in physics.) Let H be its completion. For $x \in A$, let $\pi(x)$ be the multiplication operator $y \mapsto xy$ on A. This is extended to a bounded linear operator on H and we still denote the extension by $\pi(x)$. Then π is a $*$-homomorphism from A into $B(H)$. The norm closure of $\pi(A)$ is a C^*-algebra called the type 2^∞-UHF algebra or the CAR algebra. (The abbreviations UHF and CAR stand for "Uniformly Hyperfinite" and "Canonical Anticommutation Relations", respectively.) The closure M of $\pi(A)$ in the strong operator topology is a factor and it is called the *hyperfinite type II_1 factor*. (Here the name "hyperfinite" means that we have an increasing union of finite dimensional von Neumann algebras which is dense in the strong operator topology. A hyperfinite type II_1 factor is unique up to isomorphism. Sometimes, the terminology *AFD*, standing for "approximately finite dimensional", is used instead of "hyperfinite".) The Fields medal theorem of Connes [5] gives an intrinsic characterization of the hyperfinite II_1 factor.

The linear functional tr is extended to M and satisfies the following properties.

1. We have $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for $x, y \in M$.
2. We have $\mathrm{tr}(x^*x) \geq 0$ for $x \in M$ and if $\mathrm{tr}(x^*x) = 0$, then we have $x = 0$.
3. We have $\mathrm{tr}(I) = 1$.

If an infinite dimensional von Neumann algebra has a linear functional tr satisfying the above three conditions, then it is called a type II_1 factor. Such a linear functional is unique on each type II_1 factor and called a *trace*. There are many type II_1 factors which are not hyperfinite.

A type II_∞ factor is a tensor product of a type II_1 factor and $B(H)$ for an infinite dimensional Hilbert space H.

Definition 2.8 Two projections p, q in a von Neumann algebra are said to be *equivalent* if we have u in the von Neumann algebra satisfying $p = uu^*$ and $q = u^*u$.

If uu^* is a projection, then u^*u is also automatically a projection, and such u is called a *partial isometry*.

Definition 2.9 A factor is said to be of type III if any two non-zero projections in it are equivalent and it is not isomorphic to $\mathbb{C}$.

This definition is different from the usual definition of a type III factor, but means the same condition since we consider only separable Hilbert spaces.

Two equivalent projections are analogous to two sets having the same cardinality in set theory. Then the property analogous to the above in set theory would be that any two non-empty subsets have the same cardinality for a set which is not a singleton or the empty set. Such a condition is clearly impossible in set theory. Still, based on this analogy, we interpret that the above property for a type III factor manifests a very high level of infiniteness. Because of this analogy, a type III factor is also called *purely infinite*.

The following is an example of a type III factor.

Example 2.10 Fix λ with $0 < \lambda < 1$ and set $\phi_\lambda : M_2(\mathbb{C}) \to \mathbb{C}$ by

$$\phi_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{a}{1+\lambda} + \frac{d\lambda}{1+\lambda}.$$

Let A be the same as in Example 2.7. The linear functionals $\phi_\lambda \otimes \cdots \otimes \phi_\lambda$ on $M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})$ are compatible with the embedding, so $\phi^\lambda = \bigotimes \phi_\lambda$ is well-defined on A. We set the inner product on A by $(x, y) = \phi^\lambda(y^*x)$ and set H be its completion. Let $\pi(x)$ be the left multiplication of x on A, then it is extended to a bounded linear operator on H again. The extension is still denoted by $\pi(x)$. The norm closure of $\pi(A)$ is isomorphic to the 2^∞ UHF algebra in Example 2.7. The closure M of $\pi(A)$ in the strong operator topology is a type III factor, and we have non-isomorphic von Neumann algebras for different values of λ. They are called the *Powers factors*.

It is non-trivial that Powers factors are of type III. Here we give a rough idea why this should be the case. On the one hand, two equivalent projections are regarded as "having the same size". On the other hand, now the functional ϕ^λ is also involved in measuring the size of projections. The two projections

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are equivalent, but have different "sizes" according to ϕ^λ. Because of this incompatibility, we do not have a consistent way of measuring sizes of projections, and it ends up that all nonzero projections are of "the same size" in the sense of equivalence.

Connes has refined the class of type III factors into those of type III_λ factors with $0 \le \lambda \le 1$. The Powers factors as above are of type III_λ with $0 < \lambda < 1$. If M and N are the Powers factors of type III_λ and III_μ, respectively, and $\log\lambda \log\mu$ is irrational, then $M \otimes N$ is a factor of type III_1. The isomorphism class of $M \otimes N$ does not depend on λ and μ as long as $\log\lambda/\log\mu$ is irrational, and this factor is called the *Araki-Woods factor* of type III_1. The factor which appears in conformal field theory is always this one. The Powers and Araki-Woods factors are hyperfinite. There are many type III factors which are not hyperfinite, but they do not appear in conformal field theory.

2.3 Dimensions and modules

First consider a trivial example of a factor, $M_2(\mathbb{C})$. We would like to find the "most natural" Hilbert space on which $M_2(\mathbb{C})$ acts. One might think it is clearly $\mathbb{C}^2$, but from our viewpoint of infinite dimensional operator algebras, it is not the right answer. Instead, let $M_2(\mathbb{C})$ act on itself by the left multiplication and we put a Hilbert space structure on $M_2(\mathbb{C})$ so that a natural system $\{e_{ij}\}$ of matrix units gives an orthonormal basis. Then the commutant of the left multiplication of $M_2(\mathbb{C})$ is exactly the right multiplication of $M_2(\mathbb{C})$ and thus the left and right multiplications are now symmetric. This is the *natural* representation from our viewpoint, and we would like to consider its infinite dimensional analogue.

Let M be a type II_1 factor with tr. Put an inner product on M by $(x, y) = \mathrm{tr}(y^*x)$ and denote its completion by $L^2(M)$. The left and right multiplications by an element of M on M extend to bounded linear operators on $L^2(M)$. We say that $L^2(M)$ is a left M-module and also a right M-module.

Let p be a projection in M. Then $L^2(M)p$ is naturally a left M-module. For projections $p_n \in M$, we define $\dim_M \bigoplus_n L^2(M)p_n = \sum_n \mathrm{tr}(p_n)$. Then it turns out that any left M-module H is unitarily equivalent to this form and this number $\dim_M H \in [0, \infty]$ is well-defined. It is called the *dimension* of a left M-module

H, and is a complete invariant up to unitary equivalence. Note that we have $\dim_M L^2(M) = 1$.

For a type III factor M, any two nonzero left M-modules are unitarily equivalent.

In this sense, representation theory of a type II_1 factor is dictated by a single number, the dimension, and that of a type III factor is trivial. Note that a left module of a type II or III factor is never irreducible.

2.4 Subfactors

Let M be a type II_1 factor with tr. Suppose N is a von Neumann subalgebra of M and N is also a factor of type II_1. We say $N \subset M$ is a *subfactor*. (The unit of N is assumed to be the same as that of M.) Our general reference for subfactor theory is [10]. The Hilbert space $L^2(M)$ is a left M-module, but it is also a left N-module and we have $\dim_N L^2(M)$. This number is called the *index* of the subfactor and denoted by $[M : N]$. The index value is in $[1, \infty]$. The celebrated theorem of Jones [17] is as follows.

Theorem 2.11 *The set of the index values of subfactors is equal to*

$$\{4\cos^2 \frac{\pi}{n} \mid n = 3, 4, 5, \dots\} \cup [4, \infty].$$

One is often interested in the case M is hyperfinite, when N is automatically hyperfinite. It is often assumed that the index value is finite. A subfactor $N \subset M$ is said to be *irreducible* if we have $N' \cap M = \mathbb{C}$. Irreducibility of a subfactor is also often assumed.

A subfactor is an analogue of an inclusion $L^\infty(X, \mathcal{B}_1, \mu) \subset L^\infty(X, \mathcal{B}_2, \mu)$ of commutative von Neumann algebras where $\mathcal{B}_1$ is a σ-subalgebra of $\mathcal{B}_2$ on the space X and μ is a probability measure. That is, a smaller commutative von Neumann algebra means that we have less measurable sets. For $f \in L^\infty(X, \mathcal{B}_2, \mu)$, we regard it as an element in $L^2(X, \mathcal{B}_2, \mu)$ and apply the orthogonal projection P onto $L^2(X, \mathcal{B}_1, \mu)$. Then Pf is in $L^\infty(X, \mathcal{B}_1, \mu)$, and this map from $L^\infty(X, \mathcal{B}_2, \mu)$ onto $L^\infty(X, \mathcal{B}_1, \mu)$ is called a *conditional expectation*. For a subfactor $N \subset M$ of type II_1, we have a similar map $E : M \to N$ satisfying the following properties.

1. $E(x^*x) \geq 0$ for all $x \in M$.
2. $E(x) = x$ for all $x \in N$.
3. $\mathrm{tr}(xy) = \mathrm{tr}(E(x)y)$ for all $x \in M$, $y \in N$.
4. $E(axb) = aE(x)b$ for all $x \in M$, $a, b \in N$.
5. $E(x^*) = E(x)^*$ for all $x \in M$.
6. $\|E(x)\| \leq \|x\|$ for all $x \in M$.

This map E is also called the *conditional expectation* from M onto N. Actually, properties 1, 4 and 5 follow from 2 and 6.

Kosaki [24] extended the definition of the index of a subfactor to the index of a subfactor of type III. Many results on indices of type III factors are parallel to those of type II factors.

2.5 Bimodules and relative tensor products

Let M be a type II factor with tr. The Hilbert space $L^2(M)$ is a left M-module and a right M-module. Furthermore, the left action of M and the right action of M commute, so this is an M-M bimodule. We consider a general M-N bimodule ${}_M H_N$ for type II_1 factors M and N. For a bimodule ${}_M H_N$, we have $\dim H_N$ defined in a similar way to the definition of $\dim_M H$. If we have $\dim_M H \dim H_N < \infty$, we say that the bimodule is of finite type. We consider only bimodules of finite type.

Let M, N, P be type II_1 factors and consider a general M-N bimodule ${}_M H_N$ and an N-P bimodule ${}_N K_P$. Then we can define a *relative tensor product* ${}_M H \otimes_N K_P$, which is an M-P bimodule. This is again of finite type. We have

$${}_M L^2(M) \otimes_M H_N \cong {}_M H \otimes_N L^2(N)_N \cong {}_M H_N.$$

For an M-N bimodule ${}_M H_N$, we have the *contragredient* (or *conjugate*) bimodule ${}_N \bar{H}_M$. As a Hilbert space, it consists of the vectors of the form $\bar{\xi}$ with $\xi \in H$ and has operations $\overline{\xi + \eta} = \bar{\xi} + \bar{\eta}$ and $\overline{\alpha\xi} = \bar{\alpha}\bar{\xi}$. The bimodule operation is given by $x \cdot \bar{\xi} \cdot y = \overline{y^* \cdot \xi \cdot x^*}$, where $x \in N$ and $y \in M$. This is again of finite type.

For M-N bimodules ${}_M H_N$ and ${}_M K_N$, we say that a bounded linear map $T : H \to K$ is an *intertwiner* when we have $T(x\xi y) = xT(\xi)y$ for all $x \in M$, $y \in N$, $\xi \in H$. We denote the set of all the intertwiners from H to K by $\mathrm{Hom}({}_M H_N, {}_M K_N)$. We say that ${}_M H_N$ is *irreducible* if $\mathrm{Hom}({}_M H_N, {}_M H_N) = \mathbb{C}I$. We have a natural notion of a direct sum ${}_M H_N \oplus {}_M K_N$.

A bimodule ${}_M H_N$ decomposes into a finite direct sum of irreducible bimodules, because we assume ${}_M H_N$ is of finite type here.

Start with a subfactor $N \subset M$ of type II_1 with $[M : N] < \infty$. Then the N-M bimodule ${}_N L^2(M)_M$ is of finite type. The finite relative tensor products of ${}_N L^2(M)_M$ and ${}_M L^2(M)_M$ and their irreducible decompositions produce four kinds of bimodule, N-N, N-M, M-N and M-M. They are all of finite type. We have only finitely many irreducible bimodules for one of them up to isomorphisms only if we have only finitely many irreducible bimodules for all four kinds. When this finiteness condition holds, we say the subfactor $N \subset M$ is of *finite depth*. If the index is less than 4, the subfactor is automatically of finite depth.

Consider a type II_1 subfactor $N \subset M$ of finite depth and pick a representative from each of finitely many isomorphism classes of the N-N bimodules arising in the above way. For each such ${}_N X_N$, we have $\dim_N X = \dim X_N$. For such ${}_N X_N$ and ${}_N Y_N$, the relative tensor product ${}_N X \otimes_N Y_N$ is isomorphic to $\bigoplus_j^k n_j {}_N Z_{j N}$, where $\{{}_N Z_{j N}\}$ is the set of the representatives. This gives *fusion rules* and the bimodule ${}_N L^2(M)_N$ plays the role of the identity for the relative tensor product. (Note that the name "fusion" sometimes means the relative tensor product operation is commutative, but we do not assume this here.) For each ${}_N Z_{jN}$, we have k with $\overline{{}_N Z_{jN}} \cong {}_N Z_{kN}$. We also have the *Frobenius reciprocity*, $\dim \mathrm{Hom}({}_N X \otimes_N Y_N, {}_N Z_N) = \dim \mathrm{Hom}({}_N X, {}_N Z \otimes_N \bar{Y}_N)$. The N-N bimodules isomorphic to finite direct sums of these representative N-N bimodule make a *unitary fusion category*, which is an abstract axiomatization of this system of bimodules and is some kind of a *tensor category*. A basic model of unitary fusion category is that of finite dimensional unitary representations of a finite group. We recall the definitions for unitary fusion categories as follows. (See [9] for a general reference on tensor categories.)

Definition 2.12 A category $\mathcal{C}$ is called an *abelian* category over $\mathbb{C}$ is we have the following.

1. All $\mathrm{Hom}(U, V)$ are $\mathbb{C}$-vector spaces and the compositions

$$\mathrm{Hom}(V, W) \times \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, W), \quad (\phi, \psi) \mapsto \phi \circ \psi$$

are $\mathbb{C}$-bilinear, where U, V, W are objects in $\mathcal{C}$.

2. We have a zero objects 0 in $\mathcal{C}$ with $\mathrm{Hom}(0, V) = \mathrm{Hom}(V, 0) = 0$ for all objects V in $\mathcal{C}$.

3. We have finite direct sums in $\mathcal{C}$.

4. Every morphism $\phi \in \mathrm{Hom}(U, V)$ has a kernel $\ker \phi \in \mathrm{Mor}\mathcal{C}$ and a cokernel $\mathrm{coker}\, \phi \in \mathrm{Mor}\mathcal{C}$.

5. Every morphism is the composition of an epimorphism followed by a monomorphism.

6. If $\ker \phi = 0$, then we have $\phi = \ker(\mathrm{coker}\, \phi)$ and if $\mathrm{coker}\, \phi = 0$, then we have $\phi = \mathrm{coker}(\ker \phi)$.

Definition 2.13 An object U in an abelian category $\mathcal{C}$ is called *simple* if any injection $V \hookrightarrow U$ is either 0 or an isomorphism.

An abelian category $\mathcal{C}$ is called *semisimple* if any object V is isomorphic to a direct sum of simple ones, $V \cong \bigoplus_i n_i V_i$, where V_i are simple objects, n_i are multiplicities and only finitely many n_i are nonzero.

Definition 2.14 An abelian category $\mathcal{C}$ is called a *monoidal category* if we have the following.

1. A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

2. A functorial isomorphism α_{UVW} from $(U \otimes V) \otimes W$ to $U \otimes (V \otimes W)$.

3. A unit object $\mathbf{1}$ in $\mathcal{C}$ and functorial isomorphisms $\lambda_V : \mathbf{1} \otimes V \cong V$ and $\rho_V : V \otimes \mathbf{1} \cong V$.

4. If X_1 and X_2 are two expressions obtained from $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ by inserting $\mathbf{1}$'s and brackets. Then all isomorphisms composed of α's, λ's, ρ's and their inverses are equal.

5. The functor $\otimes$ is bilinear on the space of morphisms.

6. The object $\mathbf{1}$ is simple and $\mathrm{End}(\mathbf{1}) = \mathbb{C}$.

Definition 2.15 Let $\mathcal{C}$ be a monoidal category and V be an object in $\mathcal{C}$. A *right dual* to V is an object V^* with two morphisms $e_V : V^* \otimes V \to \mathbf{1}$ and $i_V : \mathbf{1} \to V \otimes V^*$ such that we have $(\mathrm{id}_V \otimes e_V)(i_V \otimes \mathrm{id}_V) = \mathrm{id}_V$ and $(e_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes i_V) = \mathrm{id}_{V^*}$.

Similarly, we define a left dual of V to be ${}^*V$ with morphisms $e'_V : V \otimes {}^*V \to \mathbf{1}$ and $i'_V : \mathbf{1} \to {}^*V \otimes V$ satisfying similar axioms.

Definition 2.16 A monoidal category is called *rigid* if every object has right and left duals.

A *tensor category* is a rigid abelian monoidal category.

A *fusion category* is a semisimple tensor category with finitely many simple objects and finite dimensional spaces of morphisms.

Definition 2.17 A fusion category $\mathcal{C}$ over $\mathbb{C}$ is said to be unitary if we have the following conditions.

1. We have a Hilbert space structure on each Hom space.

2. We have a contravariant endofunctor $*$ on $\mathcal{C}$ which is the identity on objects.

3. We have $\|\phi\psi\| \leq \|\phi\| \, \|\psi\|$ and $\|\phi^*\phi\| = \|\phi\|^2$ for each morphism ϕ, ψ where ϕ and ψ are composable.

4. We have $(\phi \otimes \psi)^* = \phi^* \otimes \psi^*$ for each morphism ϕ, ψ.

5. All structure isomorphisms for simple objects are unitary.

For any such N-N bimodule ${}_N H_N$, we automatically have $\dim_N H = \dim H_N$. The index value of a subfactor of finite depth is known to be a cyclotomic integer automatically. Conversely, we have the following theorem.

Theorem 2.18 *Any abstract unitary fusion category is realized as that of N-N bimodules arising from some (not necessarily irreducible) subfactor $N \subset M$ with finite index and finite depth, where N and M are hyperfinite type II_1 factors.*

2.6 Classification of subfactors with small indices

Popa's celebrated classification theorem [28] say that if a hyperfinite type II_1 subfactor $N \subset M$ has a finite depth, then the finitely many irreducible bimodules of the four kinds and the intertwiners between their tensor products contain complete information on the subfactor and recover $N \subset M$.

Classification of subfactors up to index 4 was announced in [27] as follows.

Theorem 2.19 *The hyperfinite II_1 subfactors with index less than 4 are labelled with the Dynkin diagrams A_n, D_{2n}, E_6 and E_8. Subfactors corresponding to A_n and D_{2n} are unique and those corresponding to E_6 and E_8 have two isomorphism classes each.*

Recently, we have classification of subfactors with finite depth up to index 5. See [18] for details. Many of them are related to conformal field theory and quantum groups, but we see some exotic examples which have been so far unrelated to them. Up to index 5, we have three such exotic subfactors, the Haagerup subfactor [1], the Asaeda-Haagerup subfactor [1] and the extended Haagerup subfactor.

This is a very important active topic of the current research, but we refrain from going into details here.

2.7 Bimodules and endomorphisms

In this section, M is a type III factor. We present another formulation of the bimodule theory which is more useful in conformal field theory.

We can also define $L^2(M)$ as a completion of M with respect to some inner product arising from some positive linear functional on M. Then the left action of M is defined usually, and we can also define the right action of M on $L^2(M)$ using the modular conjugation in the Tomita-Takesaki theory. We then have an M-M bimodule ${}_M L^2(M)_M$, and the commutant of the left action of M is exactly the right action of M.

Consider an M-M bimodule H. The left actions of M on H and $L^2(M)$ are unitarily equivalent since M is a type III factor. So by changing H within the equivalence class of left M-modules, we may and do assume that $H = L^2(M)$ and the left actions of M on H and $L^2(M)$ are the same. Now consider the right action of M on $H = L^2(M)$. It must commute with the left action of M, but this commutant is exactly the right action of M on $L^2(M)$, so this means that a general right action of M on H is given by a homomorphism of M into M, that is, an endomorphism of M. (We consider only unital homomorphisms and endomorphisms in this text.) Conversely, if we have an endomorphism λ of M, then we can define an M-M bimodule $L^2(M)$ with the standard left action and the right action given by $x \cdot \xi \cdot y = x\xi\lambda(y)$. In this way, considering bimodules and considering endomorphisms are the same. We now see the corresponding notions of various ones in the setting of bimodules. We write $\mathrm{End}(M)$ for the set of all endomorphisms of M.

Two endomorphisms λ_1 and λ_2 of M are said to be unitarily equivalent if we have a unitary u with $\mathrm{Ad}(u) \cdot \lambda_1 = \lambda_2$. The unitary equivalence of endomorphisms corresponds to the isomorphism of bimodules. A unitary equivalence class of endomorphisms is called a *sector*. This name comes from *superselection sectors* which appear later in this text. We write $[\lambda]$ for the sector of λ.

For two endomorphisms λ_1 and λ_2 of M, we define the direct sum $\lambda_1 \oplus \lambda_2$ as follows. Since M is a factor of type III, we have isometries $V_1, V_2 \in M$ with $V_1V_1^* + V_2V_2^* = I$. Then we set $(\lambda_1 \oplus \lambda_2)(x) = V_1\lambda_1(x)V_1^* + V_2\lambda_2(x)V_2^*$. The unitary equivalence class of $\lambda_1 \oplus \lambda_2$ is well-defined, and this direct sum of endomorphisms corresponds to the direct sum of bimodules.

An intertwiner in the setting of endomorphisms is given by

$$\mathrm{Hom}(\lambda_1, \lambda_2) = \{T \in M \mid T\lambda_1(x) = \lambda_2(x)T \text{ for all } x \in M\}.$$

For two endomorphisms λ_1, λ_2, we set $\langle \lambda_1, \lambda_2 \rangle = \dim \mathrm{Hom}(\lambda_1, \lambda_2)$.

The relative tensor product of bimodules corresponds to composition of endomorphisms. The contragredient bimodule corresponds to the *conjugate endomorphism*. The conjugate endomorphism of λ is denoted by $\bar{\lambda}$ and it is well-defined *only* up to unitary equivalence. The conjugate endomorphism is also given using the *canonical endomorphism* in [25, 26] arising from the modular conjugation in the Tomita-Takesaki theory. The canonical endomorphism for a subfactor $N \subset M$ corresponds to the bimodule ${}_M L^2(M) \otimes_N L^2(M)_M$. The dual canonical endomorphism for a subfactor $N \subset M$ is an endomorphism of N corresponding to the bimodule ${}_N M_N$.

An endomorphism λ of M is said to be *irreducible* if $\lambda(M)' \cap M = \mathbb{C}$. This corresponds to irreducibility if bimodules. The *index* of λ is the index $[M : \lambda(M)]$. We set the *dimension* of λ to be $[M : \lambda(M)]^{12}$ and write $d(\lambda)$ or d_λ. Note that an endomorphism with dimension 1 is an automorphism. We have $d(\lambda_1 \oplus \lambda_2) = d(\lambda_1) + d(\lambda_2)$ and $d(\lambda_1\lambda_2) = d(\lambda_1)d(\lambda_2)$.

Suppose we have a finite set $\{\lambda_i \mid i = 0, 1, \ldots, n\}$ of endomorphisms of finite dimensions of M with λ_0 being the identity automorphism. Suppose we have the following conditions.

1. Different λ_i and λ_j are not unitarily equivalent.
2. The composition $\lambda_i\lambda_j$ is unitarily equivalent to $\bigoplus_{k=1}^n m_k\lambda_k$, where m_k is the multiplicity of λ_k.
3. For each λ_i, its conjugate $\bar{\lambda}_i$ is unitarily equivalent to some λ_j.

Then the set of endomorphisms of M unitarily equivalent to finite direct sums of $\{\lambda_i\}$ gives a unitary fusion category. This is a counterpart of the unitary fusion category of bimodules. Conversely, any abstract unitary fusion category is realized as that of endomorphisms of the type III_1 Araki-Woods factor. This is a direct consequence of Theorem 2.18.

3 Local conformal nets

We now present a precise formulation of chiral conformal field theory in the operator algebraic framework.

After introducing basic definitions, we present elementary properties and representation theory.

3.1 Definition

We now introduce the axioms for a *local conformal net* [21]. We say $I \subset S^1$ is an *interval* when it is a non-empty, connected, non-dense and open subset of S^1.

Definition 3.1 We say that a family of von Neumann algebras $\{\mathcal{A}(I)\}$ parameterized by intervals $I \subset S^1$ acting on the same Hilbert space H is a local conformal net when it satisfies the following conditions.

1. (Isotony) For two intervals $I_1 \subset I_2$, we have $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.

2. (Locality) When two intervals I_1, I_2 satisfy $I_1 \cap I_2 = \varnothing$, we have $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = 0$.

3. (Möbius covariance) We have a unitary representation U of $PSL(2,\mathbb{R})$ on H such that we have $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$ for all $g \in PSL(2,\mathbb{R})$, where g acts on S^1 as a fractional linear transformation on $\mathbb{R} \cup \{\infty\}$ and $S^1 \smallsetminus \{-1\}$ is identified with $\mathbb{R}$ through the Cayley transform $C(z) = -i(z-1)/(z+1)$.

4. (Conformal covariance) We have a projective unitary representation, still denoted by U, of $\mathrm{Diff}(S^1)$ extending the unitary representation U of $PSL(2,\mathbb{R})$ such that

$$
\begin{aligned}
U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1), \\
U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in \mathrm{Diff}(I'),
\end{aligned}
$$

   where I' is the interior of the complement of I and $\mathrm{Diff}(I')$ is the set of diffeomorphisms of S^1 which are the identity map on I.

5. (Positive energy condition) The generator of the restriction of U to the rotation subgroup of S^1, the *conformal Hamiltonian*, is positive.

6. (Existence of the vacuum vector) We have a unit vector $\Omega \in H$, called the *vacuum vector*, such that Ω is fixed by the representation U of $PSL(2,\mathbb{R})$ and $(\bigvee_{I\subset S^1} \mathcal{A}(I))\Omega$ is dense in H, where $\bigvee_{I\subset S^1} \mathcal{A}(I)$ is the von Neumann algebra generated by $\mathcal{A}(I)$'s.

7. (Irreducibility) The von Neumann algebra $\bigvee_{I\subset S^1} \mathcal{A}(I)$ is $B(H)$.

The convergence in $\mathrm{Diff}(S^1)$ is defined by uniform convergence of all the derivatives.

We say $\{\mathcal{A}(I)\}$ is a local Möbius covariant net when we drop the conformal covariance axiom.

The name "net" originally meant that the set of spacetime regions are directed with respect to inclusions, but now the set of intervals in S^1 is not directed, so this name is not appropriate, but has been widely used. Another name "pre-cosheaf" has been used in some literatures.

If the Hilbert space is 1-dimensional and all $\mathcal{A}(I)$ are just $\mathbb{C}$, all the axioms are clearly satisfied, but this example is of no interest, so we exclude this from a class of local conformal nets.

Locality comes from the fact that we have no interactions between two spacelike separated regions in the $(1+1)$-dimensional Minkowski space. Now because of the restriction procedure to two light rays, the notion of spacelike separation takes this simple form of disjointness.

The positive energy condition is our counterpart to what is called the spectrum condition in quantum field theory on the higher dimensional Minkowski space.

Irreducibility condition is equivalent to the uniqueness of the $PSL(2,\mathbb{R})$-invariant vector up to scalar, and is also equivalent to factoriality of each algebra $\mathcal{A}(I)$.

It would be better to have some easy examples here, but unfortunately, there are no easy examples one can present immediately without preparations, so we postpone examples to a later section.

We have the following consequences from the axioms.

Theorem 3.2 *(the Reeh-Schlieder theorem) For each interval $I \subset S^1$, both $\mathcal{A}(I)\Omega$ and $\mathcal{A}(I)'\Omega$ are dense in H, where $\mathcal{A}(I)'$ is the commutant of $\mathcal{A}(I)$.*

We can prove the following important result with the Tomita-Takesaki theory.

Theorem 3.3 *(the Haag duality) We have $\mathcal{A}(I)' = \mathcal{A}(I')$.*

Theorem 3.4 *Each $\mathcal{A}(I)$ is isomorphic to the Araki-Woods factor of type III_1.*

This means that in our setting the isomorphism class of each von Neumann algebra $\mathcal{A}(I)$ is unique for any interval and any local conformal net. So each $\mathcal{A}(I)$ has no information on conformal field theory, and it is the relative positions of the algebras $\mathcal{A}(I)$ that contain information of conformal field theory.

3.2 Superselection sectors and braiding

An important tool to study local conformal nets is their representation theory.

Each $\mathcal{A}(I)$ of a local conformal net acts on the Hilbert space H from the beginning by definition, but consider representations of a family $\{\mathcal{A}(I)\}$ of factors on the common Hilbert space H_π. That is, we consider a family π of representations $\pi_I : \mathcal{A}(I) \to B(H_\pi)$ such that the restriction of π_{I_2} to $\mathcal{A}(I_1)$ is equal to π_{I_1} for $I_1 \subset I_2$. Note that H_π does not have a vacuum vector in general. The original identity representation on H is called the *vacuum representation*.

For this notion of a representation, it is easy to define an irreducible representation, the direct sum of two representations and unitary equivalence of two representations. A unitary equivalence class of representations is called a *superselection sector* or a *DHR (Doplicher-Haag-Roberts) sector*.

We would like to define a notion of a tensor product of two representations. This is a non-trivial task, and an answer has been given in the Doplicher-Haag-Roberts theory [7, 8], which was originally developed for quantum field theory on the 4-dimensional Minkowski space. The Doplicher-Haag-Roberts theory adapted to conformal field theory is given as follows. (See [11, 12].)

Take a representation $\pi = \{\pi_I\}$ of a local conformal net $\{\mathcal{A}(I)\}$. Fix an arbitrary interval $I_0 \subset S^1$ and consider the representation $\pi_{I_0'}$ of $\mathcal{A}(I_0')$. Since $\mathcal{A}(I_0')$ is a type III factor, the identity representation $\mathcal{A}(I_0') \hookrightarrow B(H)$ and $\pi_{I_0'}$ are unitarily equivalent. By changing the representation π within the unitary equivalence class if necessary, we may and do assume that $H = H_\pi$ and $\pi_{I_0'}$ is the identity representation.

Take an interval I_1 with $\bar{I}_1 \subset I_0$, and then take an interval I_2 containing both I_1 and I_0'. For $x \in \mathcal{A}(I_1)$, we have $\pi_{I_2}(xy) = \pi_{I_2}(yx)$ for any $y \in \mathcal{A}(I_0')$. This implies $\pi_{I_1}(x)y = y\pi_{I_1}(x)$, and thus the image of $\mathcal{A}(I_1)$ by π_{I_0} is contained in $\mathcal{A}(I_0')' = \mathcal{A}(I_0)$. Then π_{I_0} gives an endomorphism of $\mathcal{A}(I_0)$. This endomorphism of a single algebra contains all the information about the original representation, and is called a *DHR endomorphism*.

We see that a composition of two DHR endomorphisms is again a DHR endomorphism. This defines a *tensor product* operation of two representations. This also gives a tensor product of superselection sectors.

We define the *dimension* of a DHR endomorphism λ to be the square root of the index $[\mathcal{A}(I_0) : \lambda_{I_0}(\mathcal{A}(I_0))]$ when λ is localized in I_0. This is independent of I_0. The dimension is additive and multiplicative with respect to the direct sum and the tensor product of representations.

We have a unitary tensor category consisting of endomorphisms $\mathcal{A}(I_0)$ and this gives *the* representation category of the local conformal net $\{\mathcal{A}(I)\}$. We write $\mathrm{Rep}(\mathcal{A})$ for this.

We next introduce the braiding operator $\varepsilon^\pm(\lambda,\mu)$ on this unitary tensor category.

Theorem 3.5 *Let λ, μ, ν be the localized endomorphisms of $\mathcal{A}(I_0)$. We have the following relations.*

$$\begin{aligned}
\mathrm{Ad}(\varepsilon^\pm(\lambda,\mu)) \cdot \lambda \cdot \mu &= \mu \cdot \lambda, \\
\varepsilon^\pm(\lambda.\mu) &\in \mathcal{A}(I_0), \\
\varepsilon^+(\lambda,\mu) &= \varepsilon^-(\mu,\lambda)^+, \\
\varepsilon^\pm(\lambda\cdot\mu,\nu) &= \varepsilon^\pm(\lambda,\nu)\lambda(\varepsilon^\pm(\mu,\nu)), \\
\varepsilon^\pm(\lambda,\mu\cdot\nu) &= \mu(\varepsilon^\pm(\lambda,\nu))\varepsilon^\pm(\lambda,\mu)), \\
\nu(t)\varepsilon^\pm(\lambda,\nu) &= \varepsilon^\pm(\mu,\nu)t, \quad t \in \mathrm{Hom}(\lambda,\mu), \\
t\varepsilon^\pm(\nu,\lambda) &= \varepsilon^\pm(\nu,\mu)\nu(t), \quad t \in \mathrm{Hom}(\lambda,\mu).
\end{aligned}$$

The last two identities imply the following *braiding fusion equations*.

Corollary 3.6 *Let λ, μ, ν, ρ be the localized endomorphisms of $\mathcal{A}(I_0)$. We have the following identities for $s \in \mathrm{Hom}(\lambda\cdot\mu,\nu)$.*

$$\begin{aligned}
\rho(s)\varepsilon^\pm(\lambda,\rho)\lambda(\varepsilon^\pm(\mu,\rho)) &= \varepsilon^\pm(\nu,\rho)s, \\
s\lambda(\varepsilon^\pm(\rho,\mu))\varepsilon^\pm(\rho,\lambda) &= \varepsilon^\pm(\rho,\nu)\rho(s).
\end{aligned}$$

In this way, DHR endomorphisms localized in I_0 gives a unitary braided tensor category of endomorphisms of $\mathcal{A}(I_0)$ in the following sense.

Definition 3.7 Let $\mathcal{C}$ be a monoidal category with functorial isomorphisms $\sigma_{VW} : V \otimes W \to W \otimes V$ for all objects V, W in $\mathcal{C}$.

For given objects $V_1, V_2, \ldots, V_n$ in $\mathcal{C}$, we consider all expressions of the form

$$((V_{i_1} \otimes V_{i_2}) \otimes (\mathbf{1} \otimes V_{i_3})) \otimes \cdots \otimes V_{i_n}$$

obtained from $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n}$ by inserting some $mathbf1$'s and brackets. where $(i_1, i_2, \ldots, i_n)$ is a permutation of $\{1, 2, \ldots, n\}$. To any composition of α's, λ's, ρ's, σ's and their inverses acting on the element in the above tensor product, we assign an element of the braid group B_n with the standard generators b_i $(i = 1, 2, \ldots, n-1)$ satisfying $b_i b_j = b_j b_i$ for $|i-j| > 1$ and $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ as follows. To α, λ and ρ, we assign 1 and to $\sigma_{V_{i_k} V_{i_{k+1}}}$ the generator b_k.

The category $\mathcal{C}$ is called a *braided tensor category* if for any two expressions X_1, X_2 of the above form and any isomorphism $\phi : X_1 \to X_2$ obtained by composing α's, λ's, ρ's, σ's and their inverses, ϕ depends only on its image in the braid group B_n.

If the braiding is *non-degenerate* in some appropriate sense, we say that the tensor category is a unitary *modular tensor category*. A modular tensor category produces a 3-dimenaional topological quantum field theory [29]. We can the define the S- and T-matrices naturally from a unitary representation of $SL(2, \mathbb{Z})$, where the dimension of the representation is the number of irreducible representations up to unitary equivalence.

In this case, we have the celebrated *Verlinde formula*,

$$N_{\lambda\mu}^{\nu} = \sum_{\sigma} \frac{S_{\lambda,\sigma} S_{\mu,\sigma} S^*_{\nu,\sigma}}{S_{0,\sigma}}, \tag{1}$$

where the nonnegative integer $N_{\lambda\mu}^{\nu}$ is determined by the fusion rules $\lambda \cdot \mu = \sum_{\nu} N_{\lambda\mu}^{\nu} \nu$.

3.3 Complete rationality and modular tensor categories

A unitary braided tensor category of representations of a local conformal net is similar to that of those of a quantum group at a root of unity. In such a representation theory, it is important to consider a case where we have only finitely many irreducible representations up to unitary equivalence. Such finiteness is often called *rationality*. This name comes from the fact that such finiteness of representation theory gives rationality of various parameters in conformal field theory. Based on this, we introduce the following notion.

Definition 3.8 Let $\{\mathcal{A}(I)\}$ be a local conformal net. Split the circle S^1 into four intervals and label them I_1, I_2, I_3, I_4 in the clockwise order. If the subfactor $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$ has a finite index, we say that the local conformal net $\{\mathcal{A}(I)\}$ is *completely rational*.

The reason we call this complete rationality comes from the following theorem [23, Theorem 33, Corollary 37].

Theorem 3.9 *When a local conformal net is completely rational, then it has only finitely many irreducible representations up to unitary equivalence, and all of them have finite dimensions. When this holds, the unitary braided tensor category of finite dimensional representations of* $\{\mathcal{A}(I)\}$ *is a unitary modular tensor category and the index of the above subfactor* $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$ *is equal to the square sum of the dimensions of the irreducible representations.*

We have finite dimensionality of the irreducible representations and this is why we have added the word "completely". Note that it is difficult in general to know all the irreducible representations, but the above theorem gives information on representations from a subfactor defined in the vacuum representation.

We call the index of the above subfactor $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$ the *μ-index* of $\{\mathcal{A}(I)\}$. This index is independent of the choice of I_1, I_2, I_3, I_4.

The above theorem implies that for a local conformal net, all of its irreducible representations are unitarily equivalent to the vacuum representation if and only if the local conformal net has μ-index 1, since the vacuum representation has dimension 1. We call such a local conformal net *holomorphic*. This name comes from holomorphicity of the partition function of a full conformal field theory.

3.4 Examples and construction methods

We now discuss how to construct local conformal nets. One way to construct a local conformal net is from a Kac-Moody Lie algebra, but from our viewpoint, it is easier to use a loop group for a compact Lie group. Consider a connected and simply connected Lie group, say, $SU(N)$. Let $L(SU(N))$ be the set of all the C^∞-maps from S^1 to $SU(N)$. We fix a positive integer k called a *level*. Then we have finitely many irreducible projective unitary representations of $L(SU(N))$ called *positive energy representations* at level k. We have one distinguished representation, called the *vacuum representation*, among them. For each interval $I \subset S^1$, we denote the set of C^∞-maps from S^1 to $SU(N)$ such that the image outside the interval I is always the identity matrix by $L_I(SU(N))$. Then setting $\mathcal{A}(I)$ to be the von Neumann algebra generated by the image of $L_I(SU(N))$ by the vacuum representation, we have a local conformal net $\{\mathcal{A}(I)\}$, which is labelled as $SU(N)_k$. (See [31], [14] for details.) A similar construction for other Lie groups has been done. These examples correspond to the so-called *Wess-Zumino-Witten models*, and this name is also often attached to these local conformal nets.

Another construction of a local conformal net is from a lattice Λ in the Euclidean space $\mathbb{R}^n$, that is, an additive subgroup of $\mathbb{R}^n$ which is isomorphic to $\mathbb{Z}^n$ and spans $\mathbb{R}^n$ linearly. A lattice is called *even* when we have $(x, y) \in \mathbb{Z}$ and $(x, x) \in 2\mathbb{Z}$ for the inner products of $x, y \in \Lambda$. One obtains a local conformal net from an even lattice Λ. This is like a loop group construction for $\mathbb{R}^n/\Lambda$. The local conformal nets arising from even lattices are also completely rational. Let $\Lambda^* = \{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in \Lambda\}$, the dual lattice of Λ. Then the irreducible representations of the local conformal net arising from Λ are labelled with the elements of Λ^*/Λ and all have dimension 1. It is holomorphic if and only if we have $\Lambda^* = \Lambda$.

Another construction of a local conformal net is from a vertex operator algebra. We see this construction in the next Chapter.

We next show methods to obtain new local conformal nets from known ones.

Example 3.10 For two local conformal nets $\{\mathcal{A}(I)\}$ and $\{\mathcal{B}(I)\}$, we construct a new one $\{\mathcal{A}(I) \otimes \mathcal{B}(I)\}$. This is called the *tensor product* of local conformal nets. Both the Hilbert space and the vacuum vector of the tensor product of local conformal nets are those of the tensor products. Each irreducible representation of the tensor product local conformal net is a tensor product of two irreducible representations of the two local conformal nets, up to unitary equivalence. That is, each finite dimensional representation of $\{\mathcal{A}(I) \otimes \mathcal{B}(I)\}$ is of the form $\lambda \otimes \mu$, where λ and μ are finite dimensional representations of $\{\mathcal{A}(I)\}$ and $\{\mathcal{B}(I)\}$, respectively. We also have $\mathrm{Hom}(\lambda_1 \otimes \mu_1, \lambda_2 \otimes \mu_2) = \mathrm{Hom}(\lambda_1, \lambda_2) \otimes \mathrm{Hom}(\mu_1, \mu_2)$. This representation category of $\{\mathcal{A}(I) \otimes \mathcal{B}(I)\}$ is written as $\mathrm{Rep}(\mathcal{A}) \boxtimes \mathrm{Rep}(\mathcal{B})$ and called the *Deligne product* of $\mathrm{Rep}(\mathcal{A})$ and $\mathrm{Rep}(\mathcal{B})$.

Example 3.11 The next construction is called the *simple current extension*. This is an extension of a local conformal net $\{\mathcal{A}(I)\}$ with something similar to a semi-direct product with a group. (See Example 4.20 for the initial appearance of this type of construction.) Suppose some irreducible representations of $\{\mathcal{A}(I)\}$ have dimension 1 and they are closed under the conjugation and the tensor product. If they further have all statistical phases 1, then they make a group of DHR *auto*morphisms. An automorphism used in this construction is called a *simple current* in physics literatures and this is the source of the name of the construction. This method also gives a realization of local conformal nets arising from even lattices as follows.

Example 3.12 The next one is called the *orbifold construction*. An automorphism of a local conformal net $\{\mathcal{A}(I)\}$ on H is a unitary operator U on H satisfying $U\mathcal{A}(I)U^* = \mathcal{A}(I)$ for all intervals I and $U\Omega = \Omega$. We then consider a group G of automorphisms of a local conformal net $\{\mathcal{A}(I)\}$ and define a subnet by $\mathcal{B}(I) = \{x \in \mathcal{A}(I) \mid UxU^* = x, U \in G\}$. Replacing H with the closure of $\mathcal{B}(I)\Omega$, which is independent of I, we obtain a new local conformal net $\{\mathcal{B}(I)\}$. This construction is called the *orbifold construction*. (See Example 4.21 for the initial appearance of this type of construction.) We usually consider a finite group G.

Another construction is called the *coset construction*. Suppose we have two local conformal nets $\{\mathcal{A}(I)\}$, $\{\mathcal{B}(I)\}$ where the latter is a subnet of the former. Then the family of von Neumann algebras $\mathcal{A}(I)' \cap \mathcal{B}(I)$ on the Hilbert space $\overline{(\mathcal{A}(I)' \cap \mathcal{B}(I))\Omega}$ gives a new local conformal net. (This Hilbert space is again independent of I.) This construction is called the coset construction.

Also when we perform the coset construction for $\{\mathcal{A}(I) \subset \mathcal{B}(I)\}$ with completely rationality of $\{\mathcal{B}(I)\}$ and finiteness of the index $[\mathcal{B}(I) : \mathcal{A}(I) \vee (\mathcal{A}(I)' \cap \mathcal{B}(I))]$, we have complete rationality of $\{\mathcal{A}(I)' \cap \mathcal{B}(I)\}$.

4 Vertex operator algebras

We have a different axiomatization of chiral conformal field theory from a local conformal net and it is a theory of *vertex operator algebras*. It is a direct axiomatization of Wightman fields on the circle S^1. In physics literatures, certain operator-valued distributions are called vertex operators and this is the origin of the name "vertex operator algebra". We explain this theory in comparison to that of local conformal nets. We emphasize relations to local conformal nets rather than a general theory of vertex operator algebras.

A certain amount of the theory has been devoted to a single example called the *Moonshine vertex operator algebra*, so we explain the background of the Moonshine conjecture for which it was constructed.

Among finite groups, finite simple groups are clearly fundamental. Today we have a complete list of finite simple groups as follows.

1. Cyclic groups of prime order.
2. Alternating groups of degree 5 or higher.
3. 16 series of groups of Lie type over finite fields.
4. 26 sporadic finite simple groups.

The third class consists of matrix groups such as $PSL(n, \mathbb{F}_q)$. The last class consists of exceptional structures, and the first ones were found by Mathieu in the 19th century. The largest group among the 26 groups in terms of the order is called the *Monster group*, and its order is

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71,$$

which is approximately 8×10^{53}. This group was first constructed in [15] as the automorphism group of some commutative, but nonassociative algebra of 196884 dimensions. From the beginning, it has been known that the smallest dimension of a non-trivial irreducible representation of the Monster group is 196883.

Now we turn to a different topic of the classical *j-function*. This is a function of a complex number τ with $\mathrm{Im}\,\tau > 0$ given as follows.

$$\begin{aligned} j(\tau) &= \frac{(1 + 240\sum_{n>0} \sigma_3(n) q^n)^3}{q\prod_{n>0}(1-q^n)^{24}} \\ &= q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots, \end{aligned}$$

where $\sigma_3(n)$ is a sum of the cubes of the divisors of n and we set $q = \exp(2\pi i\tau)$.

This function has modular invariance property

$$j(\tau) = j\left(\frac{a\tau + b}{c\tau + d}\right),$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

and this property and the condition that the Laurent series of q start with q^{-1} determine the j-function uniquely except for the constant term. The constant term 744 has been chosen for a historic reason and this has no significance, so we set $J(\tau) = j(\tau) - 744$ and use this from now on.

McKay noticed that the first non-trivial coefficient of the Laurent expansion of the J-function satisfies the equality $196884 = 196883 + 1$, where 1 the dimension of the trivial representation of the Monster group and 196883 is the smallest dimension of its non-trivial representation. People suspected it is simply a coincidence, but it has turned out that all the coefficients of the Laurent expansion of the J-function with small exponents are linear combinations of the dimensions of irreducible representations of the Monster group with "small" positive integer coefficients. (We have 1 as the dimension of the trivial representation, so it is trivial that any positive integer is a sum of the dimensions of irreducible representations of the Monster group, but it is highly non-trivial that we have small multiplicities.)

Based on this, Conway-Norton [6] formulated what is called the *Moonshine conjecture* today, which has been proved by Borcherds [3].

Conjecture 4.1 *1. We have some graded infinite dimensional $\mathbb{C}$-vector space $V = \bigoplus_{n=0}^{\infty} V_n$ $(\dim V_n < \infty)$ with some natural algebraic structure and its automorphism group is the Monster group.*

2. Each element g of the Monster group acts on each V_n linearly. The Laurent series

$$\sum_{n=0}^{\infty} (\mathrm{Tr}\, g|_{V_n}) q^{n-1}$$

arising from the trace value of the g-action on V_n is a classical function called a Hauptmodul corresponding to a genus 0 subgroup of $SL(2, \mathbb{R})$. (The case g is the identity element is the J-function.)

The above Laurent series is called the McKay-Thompson series. The first statement is vague since it does not specify the "natural algebraic structure", but Frenkel-Lepowsky-Meurman [13] introduced the axioms for *vertex operator algebras* and constructed an example $V^\natural$, called the Moonshine vertex operator algebra, corresponding to the first statement of the above Conjecture. This was the starting point of the entire theory.

4.1 Basic definitions

There are various, slightly different versions of the definition of vertex operator algebras, so we fix our definition here. We follow [4].

Let V be a $\mathbb{C}$-vector space. We say that a formal series $a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ with coefficients $a_{(n)} \in \mathrm{End}(V)$ is a *field* on V, if for any $b \in V$, we have $a_{(n)} b = 0$ for all sufficiently large n.

Definition 4.2 A $\mathbb{C}$-vector space V is a *vertex algebra* if we have the following properties.

1. (State-field correspondence) For each $a \in V$, we have a field $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ on V.

2. (Translation covariance) We have a linear map $T \in \mathrm{End}(V)$ such that we have $[T, Y(a,z)] = \frac{d}{dz}Y(a,z)$ for all $a \in V$.

3. (Existence of the vacuum vector) We have a vector $\Omega \in V$ with $T\Omega = 0$, $Y(\Omega, z) = \mathrm{id}_V$, $a_{(-1)}\Omega = a$.

4. (Locality) For all $a, b \in V$, we have $(z-w)^N[Y(a,z), Y(b,w)] = 0$ for a sufficiently large integer N.

We then call $Y(a,z)$ a *vertex operator*.

A vertex operator is an algebraic version of the Fourier expansion of an operator-valued distribution on the circle. The state-field correspondence means that any vector in V gives an operator-valued distribution. The locality axiom is one representation of the idea that $Y(a,z)$ and $Y(b,w)$ commute for $z \neq w$. (Recall that a distribution T on $\mathbb{R}$ has $\mathrm{supp}\, T \subset \{0\}$ if and only if there exists a positive integer N with $x^N T = 0$.)

The following *Borcherds identity* is a consequence of the above axioms, where $a, b, c \in V$ and $m, n, k \in \mathbb{Z}$.

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}c \\
&= \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} a_{(m+n-j)} b_{(k+j)} c - \sum_{j=0}^{\infty} (-1)^{j+n} \binom{n}{j} b_{(n+k-j)} a_{(m+j)} c.
\end{aligned}
$$

Definition 4.3 We say a linear subspace $W \subset V$ is a *vertex subalgebra* if we have $\Omega \in W$ and $a_{(n)}b \in W$ for all $a, b \in W$ and $n \in \mathbb{Z}$. (In this case, W is automatically T-invariant.) We say a linear subspace $J \subset V$ is an *ideal* if it is T-invariant and we have $a_{(n)}b \in J$ for all $a \in V$, $b \in J$ and $n \in \mathbb{Z}$. A vertex algebra is said to be *simple* if any ideal in V is either 0 or V. A *(antilinear) homomorphism* from a vertex algebra V to a vertex algebra W is an (anti)linear map ϕ satisfying $\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b)$ for all $a, b \in V$ and $n \in \mathbb{Z}$. We similarly define an *automorphism*.

If J is an ideal of V, we also have $a_{(n)}b \in J$ for all $a \in J$, $b \in V$ and $n \in \mathbb{Z}$.

We next introduce conformal symmetry in this context.

Definition 4.4 Let V be a $\mathbb{C}$-vector space and $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ be a field on V. If the endomorphisms L_n satisfy the Virasoro algebra relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{(m^3-m)\delta_{m+n,0}}{12}c,$$

with central charge $c \in \mathbb{C}$, then we say $L(z)$ is a *Virasoro field*. If V is a vertex algebra and $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is a Virasoro field, then we say $\omega \in V$ is a

Virasoro vector. A Virasoro vector ω is called a *conformal vector* if $L_{-1} = T$ and L_0 is diagonalizable on V. (The latter means that V is an algebraic direct sum of the eigenspaces of L_0.) Then the corresponding vertex operator $Y(\omega, z)$ is called the *energy-momentum field* and L_0 the *conformal Hamiltonian*. A vertex algebra with a conformal vector is called a *conformal vertex algebra*. We then say V has central charge $c \in \mathbb{C}$.

Definition 4.5 A nonzero element a of a conformal vertex algebra in $\mathrm{Ker}(L_0 - \alpha)$ is said to be a *homogeneous* element of *conformal weight* $d_a = \alpha$. We then set $a_n = a_{(n+d_a-1)}$ for $n \in \mathbb{Z} - d_a$. For a sum a of homogeneous elements, we extend a_n by linearity.

Definition 4.6 A homogeneous element a in a conformal vertex algebra V and the corresponding field $Y(a, z)$ are called *quasi-primary* if $L_1 a = 0$ and *primary* if $L_n a = 0$ for all $n > 0$.

Definition 4.7 We say that a conformal vertex algebra V is of *CFT type* if we have $\mathrm{Ker}(L_0 - \alpha) \neq 0$ only for $\alpha \in \{0, 1, 2, 3, \dots\}$ and $V_0 = \mathbb{C}\Omega$.

Definition 4.8 We say that a conformal vertex algebra V is a *vertex operator algebra* if we have the following.

1. We have $V = \bigoplus_{n \in \mathbb{Z}} V_n$, where $V_n = \mathrm{Ker}(L_0 - n)$.
2. We have $V_n = 0$ for all sufficiently small n.
3. We have $\dim(V_n) < \infty$ for $n \in \mathbb{Z}$.

Definition 4.9 An *invariant bilinear form* on a vertex operator algebra V is a bilinear form $(\cdot, \cdot)$ on V satisfying

$$(Y(a, z)b, c) = (b, Y(e^{zL_1}(-z^{-2})^{L_0} a, z^{-1})c)$$

for all $a, b, c \in V$.

Definition 4.10 For a vertex operator algebra V with a conformal vector ω, an automorphism g as a vertex algebra is called a *VOA automorphism* if we have $g(\omega) = \omega$.

Definition 4.11 Let V be a vertex operator algebra and suppose we have a positive definite inner product $(\cdot \mid \cdot)$, where this is supposed to be antilinear in the first variable. We say the inner product is *normalized* if we have $(\Omega \mid \Omega) = 1$. We say that the inner product is *invariant* if there exists a VOA antilinear automorphism θ of V such that $(\theta \cdot \mid \cdot)$ is an invariant bilinear form on V. We say that θ is a *PCT operator* associated with the inner product.

If we have an invariant inner product, we automatically have $(L_n a \mid b) = (a \mid L_{-n} b)$ for $a, b \in V$ and also $V_n = 0$ for $n < 0$. The PCT operator θ is unique and we have $\theta^2 = 1$ and $(\theta a \mid \theta b) = (b \mid a)$ for all $a, b \in V$. (See [4, Section 5.1] for details.)

Definition 4.12 A *unitary* vertex operator algebra is a pair of a vertex operator algebra and a normalized invariant inner product.

A unitary vertex operator algebra is simple if and only if we have $V_0 = \mathbb{C}\Omega$.

For a unitary vertex operator algebra V, we write $\mathrm{Aut}_{(\cdot|\cdot)}(V)$ for the automorphism group fixing the inner product.

Definition 4.13 A *unitary subalgebra* W of a unitary vertex operator algebra $(V, (\cdot \mid \cdot))$ is a vertex subalgebra with $\theta W \subset W$ and $L_1 W \subset W$.

4.2 Modules and modular tensor categories

We introduce a notion of a module of a vertex operator algebra, which corresponds to a representation of a local conformal net, as follows.

Definition 4.14 Let M be a $\mathbb{C}$-vector space and suppose we have a field $Y^M(a,z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$, $a^M_{(n)} \in \mathrm{End}(M)$, on M for any $a \in V$, where the map $a \mapsto Y^M(a,z)$ is linear and V is a vertex algebra. We say M is a module over V if we have $Y^M(\Omega, z) = \mathrm{id}_M$ and the following Borcherds identity for $a, b \in V$, $c \in M$, $m, n, k \in \mathbb{Z}$.

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \begin{pmatrix} m \\ j \end{pmatrix} (a_{(n+j)}b)^M_{(m+k-j)} c \\
&= \sum_{j=0}^{\infty} (-1)^j \begin{pmatrix} n \\ j \end{pmatrix} a^M_{(m+n-j)} b^M_{(k+j)} c - \sum_{j=0}^{\infty} (-1)^{j+n} \begin{pmatrix} n \\ j \end{pmatrix} b^M_{(n+k-j)} a^M_{(m+j)} c.
\end{aligned}
$$

In this section, we consider only simple vertex operator algebras of CFT type.

The fusion rules on modules have been introduced and they give the tensor product operations on modules.

We have a natural notion of irreducible module and it is of the form $M = \bigoplus_{n=0}^{\infty} M_n$, where every M_n is finite dimensional and L_0 acts on M_n as a scalar $n+h$ for some constant h. The vertex operator algebra V itself is a module of V with $h = 0$, and if this is the only irreducible module, then we say V is *holomorphic*.

We define the formal power series, the *character* of M, by $\mathrm{ch}(M) = \sum_{n=0}^{\infty} \dim(M_n) q^{n+h-c/24}$, where c is the central charge. We introduce the following important notion.

Definition 4.15 If the quotient space $V/\{v_{(2)}w \mid v, w \in V\}$ is finite dimensional, we say that the vertex operator algebra V is *C_2-cofinite*.

It has been proved by Zhu that under the C_2-cofiniteness condition and small other conditions, we have only finitely many irreducible modules $M_1, M_2, \ldots, M_k$ up to isomorphism, their characters are absolutely convergent for $|q| < 1$, and the linear span of $\mathrm{ch}(M_1), \mathrm{ch}(M_2), \ldots, \mathrm{ch}(M_k)$ is closed under the action of $SL(2,\mathbb{Z})$ on the upper half plane through the fractional linear transformation on τ with $q = \exp(2\pi i \tau)$.

Under the C_2-cofiniteness assumption and small other assumptions, Huang further showed that the S-matrix defined by the transformation $\tau \mapsto -1/\tau$ on the characters satisfy the Verlinde formula (1) with respect to the fusion rules and the tensor category of the modules is modular.

4.3 Examples and construction methods

The construction methods of local conformal nets we have explained had been known in the context of vertex operator algebras earlier except for an extension by a Frobenius algebra.

Example 4.16 *We have a simple unitary vertex operator algebra $L(c, 0)$ with central charge c arising from a unitary representation of the Virasoro algebra with central charge c and the lowest conformal energy* 0.

Example 4.17 *Let $\mathfrak{g}$ be a simple complex Lie algebra and $V_{\mathfrak{g}_k}$ be the conformal vertex algebra generated by the unitary representation of the affine Lie algebra associated with $\mathfrak{g}$ having level k and lowest conformal energy* 0*. Then $V_{\mathfrak{g}_k}$ is a simple unitary vertex operator algebra.*

Example 4.18 *Let $\Lambda \subset \mathbb{R}^n$ be an even lattice. That is, it is isomorphic to $\mathbb{Z}^n$ as an abelian group and linearly spans the entire $\mathbb{R}^n$, and we have $(x, y) \in \mathbb{Z}$ and $(x, x) \in 2\mathbb{Z}$ for all $x, y \in \Lambda$, where $(\cdot, \cdot)$ is the standard Euclidean inner product. There is a general construction of a unitary vertex operator algebra from such a lattice and we obtain V_Λ from Λ. The central charge is the rank of Λ.*

Example 4.19 *If we have two unitary vertex operator algebras $(V, (\cdot \mid \cdot))$ and $(W, (\cdot \mid \cdot))$, then the tensor product $V \otimes W$ has a natural inner product with which we have a unitary vertex operator algebra.*

Example 4.20 *For a unitary vertex operator algebra with certain modules called* simple currents *satisfying some nice compatibility condition, we can extend the unitary vertex operator algebra. This is called a* simple current extension.

Example 4.21 *For a unitary vertex operator algebra $(V, (\cdot \mid \cdot))$ and $G \subset \mathrm{Aut}_{(\cdot|\cdot)}(V)$, the fixed point subalgebra V^G is a unitary vertex operator algebra. This is called an* orbifold subalgebra.

Example 4.22 *Let $(V, (\cdot \mid \cdot))$ be a unitary vertex operator algebra and W its subalgebra. Then*

$$W^c = \{b \in V \mid [Y(a, z), Y(b, w)] = 0 \text{ for all } a \in W\}$$

is a vertex subalgebra of V. This is called a coset subalgebra*. This is also called the* commutant *of W in V.*

4.4 Moonshine vertex operator algebras

A rough outline of the construction of the Moonshine vertex operator algebra is as follows. We have an exceptional even lattice in dimension 24 called the *Leech lattice*. It is the unique 24-dimensional even lattice Λ with $\Lambda = \Lambda^*$ and having no vectors $x \in \Lambda$ with $(x, x) = 2$. We have a corresponding unitary vertex operator algebra V_Λ. The involution $x \to -x$ on Λ induces an automorphism of V_Λ of order 2. Its fixed point vertex operator subalgebra has a non-trivial simple current extension of order 2. Taking this extension is called the *twisted orbifold construction* and we obtain $V^\natural$ with this. This is the Moonshine vertex operator algebra.

The local conformal net corresponding to the Moonshine vertex operator algebra has been constructed in [22] and its automorphism group in the operator algebraic sense is the Monster group.

4.5 Local conformal nets and vertex operator algebras

We now consider relations between vertex operator algebras and local conformal nets. Both are supposed to be mathematical axiomatizations of the same physical theory, so we might expect the two sets of axioms are equivalent in the sense that we have a canonical bijective correspondence between the mathematical objects satisfying one set of axioms and those for the other. However, both axiomatizations are broad and may contain some weird examples, so it is expected that we have to impose some more conditions in order to obtain such a nice bijective correspondence.

In principle, when we have some idea, example or construction on local conformal nets or vertex operator algebras, one can often "translate" it to the other side. For example, the local conformal net corresponding to the Moonshine vertex operator algebra has been constructed in [22] and its automorphism group in the operator algebraic sense is the Monster group. Such a translation has been done on a case-by-case basis. It is sometimes easy, sometimes difficult, and sometimes still unknown.

Here we deal with a construction of a local conformal net from a unitary vertex operator algebra with some extra nice properties.

Definition 4.23 Let $(V, (\cdot \mid \cdot))$ be a unitary vertex operator algebra. We say that $a \in V$ (or $Y(a, z)$) satisfies *energy-bounds* if we have positive integers s, k and a constant $M > 0$ such that we have

$$\|a_n b\| \leq M(|n| + 1)^s \|(L_0 + 1)^k b\|,$$

for all $b \in V$ and $n \in \mathbb{Z}$. If every $a \in V$ satisfies energy-bounds, we say V is *energy-bounded*.

We have the following proposition, which is [4, Proposition 6.1].

Proposition 4.24 *If V is generated by a family of homogeneous elements satisfying energy-bounds, then V is energy-bounded.*

Roughly speaking, we need norm estimates for $(a_n b)_m c$ from those for $a_n(b_m c)$ and $b_m(a_n c)$. This is essentially done with the Borcherds identity.

We also have the following proposition, which is [4, Proposition 6.3].

Proposition 4.25 *If V is a simple unitary vertex operator algebra generated by V_1 and $F \subset V_2$ where F is a family of quasi-primary θ-invariant Virasoro vectors, then V is energy-bounded.*

We have certain commutation relations for elements in V_1 and F, and this implies energy-bounds for them. Then the above Proposition follows from the previous one.

For a unitary vertex operator algebra $(V, (\cdot \mid \cdot))$, define a Hilbert space H by the completion of V with respect to the inner product $(\cdot \mid \cdot)$. For any $a \in V$ and $n \in \mathbb{Z}$, we regard $a_{(n)}$ as a densely defined operator on H. By the invariance of the scalar product, the operator $a_{(n)}$ has a densely defined adjoint, so it is closable. Suppose V is energy-bounded and let $f(z)$ be a smooth function on $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with Fourier coefficients

$$\hat{f}_n = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}$$

for $n \in \mathbb{Z}$. For every $a \in V$, we define the operator $Y_0(a, f)$ with domain V by

$$Y_0(a, f)b = \sum_{n \in \mathbb{Z}} \hat{f}_n a_n b$$

for $b \in V$. The convergence follows from the energy-bounds and $Y_0(a, f)$ is a densely defined operator. This is again closable. We denote by $Y(a, f)$ the closure of $Y_0(a, f)$ and call it a *smeared vertex operator*.

We define $\mathcal{A}_{(V,(\cdot|\cdot))}(I)$ to be the von Neumann algebra generated by the (possibly unbounded) operators $Y(a, f)$ with $a \in V$, $f \in C^\infty(S^1)$ and $\operatorname{supp} f \subset I$. (For a family of closed operators $\{T_i\}$, we apply the polar decomposition to each T_i and consider the von Neumann algebra generated by the partial isometry part of T_i and the spectral projections of the self-adjoint part of T_i.) The family $\{\mathcal{A}_{(V,(\cdot|\cdot))}(I)\}$ clearly satisfies isotony. We can verify that $(\bigvee_I \mathcal{A}_{(V,(\cdot|\cdot))}(I))\Omega$ is dense in H. A proof of conformal covariance is nontrivial, but can be done by studying the representations of the Virasoro algebra and $\mathrm{Diff}(S^1)$. We also have the vacuum vector Ω and the positive energy condition. However, locality is not clear at all from our construction, so we make the following definition.

Definition 4.26 We say that a unitary vertex operator algebra $(V, (\cdot \mid \cdot))$ is *strongly local* if it is energy-bounded and we have $\mathcal{A}_{(V,(\cdot|\cdot))}(I) \subset \mathcal{A}_{(V,(\cdot|\cdot))}(I')'$ for all intervals $I \subset S^1$.

Difficulty in having strong locality is seen as follows. It is well-known that if A and B are unbounded self-adjoint operators, having $AB = BA$ on a common core does not imply commutativity of the spectral projections of A and B. That is, having commutativity of spectral projections from certain algebraic commutativity relations is a nontrivial task.

A strongly local unitary vertex operator algebra produces a local conformal net through the above procedure by definition. The following is [4, Theorem 6.9].

Theorem 4.27 *Let V be a strongly local unitary vertex operator algebra and $\{\mathcal{A}_{(V,(\cdot|\cdot))}(I)\}$ the corresponding local conformal net. Then we have* $\mathrm{Aut}(\mathcal{A}_{(V,(\cdot|\cdot))}) = \mathrm{Aut}_{(\cdot|\cdot)}(V)$*. If* $\mathrm{Aut}(V)$ *is finite, then we have* $\mathrm{Aut}(\mathcal{A}_{(V,(\cdot|\cdot))}) = \mathrm{Aut}(V)$*.*

We now have the following theorem for a criterion of strong locality [4, Theorem 8.1].

Theorem 4.28 *Let V be a simple unitary energy-bounded vertex operator algebra and $F \subset V$. Suppose F contains only quasi-primary elements, F generates V and $\mathcal{A}_{(F,(\cdot|\cdot))}(I) \subset \mathcal{A}_{(F,(\cdot|\cdot))}(I')'$ for some interval I, where $\mathcal{A}_{(F,(\cdot|\cdot))}(I)$ is defined similarly to $\mathcal{A}_{(V,(\cdot|\cdot))}(I)$. We then have $\mathcal{A}_{(F,(\cdot|\cdot))}(I) = \mathcal{A}_{(V,(\cdot|\cdot))}(I)$ for all intervals I, which implies strongly locality of $\{\mathcal{A}_{(V,(\cdot|\cdot))}(I)\}$.*

From this, we can prove the following result, [4, Theorem 8.3].

Theorem 4.29 *Let V be a simple unitary vertex operator algebra generated by $V_1 \cup F$ where $F \subset V_2$ is a family of quasi-primary θ-invariant Virasoro vectors, then V is strongly local.*

We also have the following result, [4, Theorem 7.1]

Theorem 4.30 *Let V be a simple unitary strongly local vertex operator algebra and W its subalgebra. Then W is also strongly local.*

The following is [4, Corollary 8.2].

Theorem 4.31 *Let V_1, V_2 be simple unitary strongly local vertex operator algebras. Then $V_1 \otimes V_2$ is also strongly local.*

We list some examples of strongly local vertex operator algebras following [4].

Example 4.32 *The unitary vertex algebra $L(c, 0)$ is strongly local.*

Example 4.33 *Let $\mathfrak{g}$ be a complex simple Lie algebra and let $V_{\mathfrak{g}_k}$ be the corresponding level k unitary vertex operator algebra. Then $V_{\mathfrak{g}_k}$ is generated by $(V_{\mathfrak{g}_k})_1$ and hence it is strongly local.*

The following is (a part of) [4, Theorem 9.2].

Theorem 4.34 *Let V be a simple unitary strongly local vertex operator algebra and $\{\mathcal{A}_{(V,(\cdot|\cdot))}(I)\}$ be the corresponding local conformal net. Then one can recover the vertex operator algebra structure on V, which is an algebraic direct sum of the eigenspaces of the conformal Hamiltonian, from the local conformal net $\{\mathcal{A}_{(V,(\cdot|\cdot))}(I)\}$.*

This is proven by constructing the smeared vertex operators from abstract considerations using only the local conformal net $\{\mathcal{A}_{(V,(\cdot|\cdot))}(I)\}$.

We remark that a relation between local conformal nets and unitary vertex operator algebras is somehow similar to that between Lie groups and Lie algebras. The relation between loop groups and Kac-Moody Lie algebras is somewhere between the two relations.

Recently, Gui [16] has extended the above contsruction of a local conformal net from a vertex operator algebra to their representations satisfying some nice assumption. This is a great achievement and further progress along this line is expected.

References

[1] M. Asaeda and U. Haagerup, Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$, *Comm. Math. Phys.* **202** (1999), 1–63.

[2] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nucl. Phys.* **241** (1984), 333–380.

[3] R. E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, *Invent. Math.* **109** (1992), 405–444.

[4] S. Carpi, Y. Kawahigashi, R. Longo and M. Weiner, From vertex operator algebras to conformal nets and back, *Mem. Amer. Math. Soc.* **254** (2018), no. 1213, vi+85 pp.

[5] A. Connes, Classification of injective factors cases II_1, II_∞, III_λ, $\lambda \neq 1$, *Ann. of Math.* **104** (1976), 73–115.

[6] J. H. Conway and S. P. Norton, Monstrous moonshine, *Bull. London Math. Soc.* **11** (1979), 308–339.

[7] S. Doplicher, R. Haag and J. E. Roberts, Local observables and particle statistics, I. *Comm. Math. Phys.* **23** (1971), 199–230.

[8] S. Doplicher, R. Haag and J. E. Roberts, Local observables and particle statistics, II. *Comm. Math. Phys.* **35** (1974), 49–85.

[9] P. Etingof, D. Nikshych and V. Ostrik, On fusion categories, *Ann. of Math.* **162** (2005) 581–642.

[10] D. E. Evans and Y. Kawahigashi, "Quantum Symmetries on Operator Algebras", Oxford University Press, Oxford (1998).

[11] K. Fredenhagen, K.-H. Rehren and B. Schroer, Superselection sectors with braid group statistics and exchange algebras, I. *Comm. Math. Phys.* **125** (1989), 201–226;

[12] K. Fredenhagen, K.-H. Rehren and B. Schroer, Superselection sectors with braid group statistics and exchange algebras, II. *Rev. Math. Phys.* **Special issue** (1992), 113–157.

[13] I. Frenkel, J. Lepowsky and A. Meurman, "Vertex operator algebras and the Monster", Academic Press (1988).

[14] F. Gabbiani and J. Fröhlich, Operator algebras and conformal field theory. *Comm. Math. Phys.* **155** (1993), 569–640.

[15] R. L. Griess, Jr., The friendly giant, *Invent. Math.* **69** (1982), 1–102.

[16] B. Gui, Categorical extensions of conformal nets, preprint arXiv:1812.04470.

[17] V. F. R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.

[18] V. F. R. Jones, S. Morrison and N. Snyder, The classification of subfactors of index at most 5, *Bull. Amer. Math. Soc. (N.S.)* **51** (2014), 277–327.

[19] Y. Kawahigashi, Conformal field theory, tensor categories and operator algebras, *J. Phys. A* **48** (2015), 303001.

[20] Y. Kawahigashi, Conformal field theory, vertex operator algebras and operator algebras, to appear in the Proceedings of ICM-2018.

[21] Y. Kawahigashi and R. Longo, Classification of local conformal nets. Case $c < 1$, *Ann. of Math.* **160** (2004), 493–522.

[22] Y. Kawahigashi and R. Longo, Local conformal nets arising from framed vertex operator algebras, *Adv. Math.* **206** (2006), 729–751.

[23] Y. Kawahigashi, R. Longo and M. Müger, Multi-interval subfactors and modularity of representations in conformal field theory, *Comm. Math. Phys.* **219** (2001), 631–669.

[24] H. Kosaki, Extension of Jones' theory on index to arbitrary factors. *J. Funct. Anal.* **66** (1986), 123–140.

[25] R. Longo, Index of subfactors and statistics of quantum fields, I. *Comm. Math. Phys.* **126** (1989), 217–247.

[26] R. Longo, Index of subfactors and statistics of quantum fields, II. Correspondences, braid group statistics and Jones polynomial, *Comm. Math. Phys.* **130** (1990), 285–309.

[27] A. Ocneanu, Quantized group, string algebras and Galois theory for algebras, in *Operator algebras and applications, Vol. 2 (Warwick, 1987)*, (ed. D. E. Evans and M. Takesaki), London Mathematical Society Lecture Note Series **36**, Cambridge University Press, Cambridge, (1988), 119–172.

[28] S. Popa, Classification of amenable subfactors of type II, *Acta Math.* **172** (1994), 163–255.

[29] N. Reshetikhin and V. G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103** (1991), 547–597.

[30] M. Takesaki, "Theory of operator algebras. I", Springer-Verlag, Berlin (2002).

[31] A. Wassermann, Operator algebras and conformal field theory III: Fusion of positive energy representations of $SU(N)$ using bounded operators, *Invent. Math.* **133** (1998), 467–538.